Math@Iowa Analysis Qualifying Exam
Tuesday, August 19, 2025
9:00 am - 12:00 noon in Room 118 MLH

Examination Committee: Professors Ionuţ Chifan and Raúl E. Curto

Your Name:

Instructions

1. This is a closed-book exam; no books or notes are allowed. This exam includes a Prep Sheet (attached), which you may use as a reference.
2. You are not allowed to use any internet resources; use of calculators is not allowed.
3. This Qualifying Exam consists of 10 questions, broken down into two parts of 5 questions each, for a maximum score of 70 points. Please submit **a total of seven question, with at least three chosen from each Part**. Please summarize your choices in the table below, by **placing a ✓ inside the box located next to each choice**.
5. Please write your answers within the space provided for each question. If at all possible, please write your answers in ink (blue, black or red). However, if writing math using a pencil is what you always do, by all means use a pencil for this exam too.
6. **For each solution you provide, make sure to include all the details, and state correctly all the known results you use. If appropriate, feel free to refer to the attached Prep Sheet.**
7. A score in the range **63 – 70** guarantees a **Ph.D. Pass**; a score in the range **56 – 62** guarantees an **M.S. Conditional Pass**.
8. **Before you start, please write your name on the space provided in this page. When you finish, please turn in this exam booklet.**

☐	R-1A	☐	R-1B	☐	C-1A	☐	C-1B
☐	R-2A	☐	R-2B	☐	C-2A	☐	C-2B
☐	R-3A	☐	R-3B	☐	C-3A	☐	C-3B
☐	R-4A	☐	R-4B	☐	C-4A	☐	C-4B
☐	R-5A	☐	R-5B	☐	C-5A	☐	C-5B

PART I: Real Analysis

(10 points) **(R-1)**

Choose either Problem R-1A or Problem R-1B below and solve it.

Problem R-1A. Let $E := [0,1]$, $g \in L^2(E)$, and define $f : E \to \mathbb{R}$ as

$$f(x) := \int_0^x \sqrt{t} \cdot g(t)\, dt \qquad (0 \le x \le 1).$$

(a) Prove that f is absolutely continuous on E.
(b) Prove that f may fail to be Lipschitz on E; that is, find $g \in L^2(E)$ such that the associated f is not Lipschitz on E.
(c) Prove that f is Lipschitz when $g \in L^\infty(E)$.

Problem R-1B. Determine if the following statement is true or false. If true, provide a proof; if false, provide a counterexample or show in some fashion why the statement is false. In either case, feel free to use results from the Prep Sheet.

Let $\{f_n\}_{n=1}^\infty$ be a sequence of continuous functions on $[0,1]$ such that, for every $x \in [0,1]$, $f_n(x) \to 0$ as $n \to \infty$.

(True or False) Then

$$\lim_{n\to\infty} \int_0^1 f_n(x)\, dx = 0.$$

Problem R-1 cont.

(10 points) **(R-2)**

Choose either Problem R-2A or Problem R-2B below and solve it.

Problem R-2A. Let $f \in L^2(0,\infty)$ and define

$$F(x) := \frac{1}{x}\int_0^x f(t)\,dt \quad (0 < x < \infty).$$

Prove Hardy's inequality, that is,

$$\|F\|_2 \leq 2\,\|f\|_2\,.$$

(Hint: Assume first that $f \in C_c(0,\infty)$ and use integration by parts to conclude that

$$\int_0^\infty F^2(x)\,dx = -2\int_0^\infty F(x)\cdot x\cdot F'(x)\,dx.$$

Next, observe that $x \cdot F' = f - F$, and apply Hölder's inequality to $\int_0^\infty F \cdot f$.)

Problem R-2B. Let $f : \mathbb{R} \to \mathbb{R}$ be an absolutely continuous function such that $(f')^{2025}$ is integrable on $\mathbb{R}$. Show that the series

$$\sum_{n=-\infty}^{\infty} |f(n+1) - f(n)|^{2025}$$

is convergent.

Problem R-2 cont.

(10 points) **(R-3)**

Choose either Problem R-3A or Problem R-3B below and solve it.

Problem R-3A. Let f and g be two positive measurable functions on $[0,1]$ such that $fg \geq 1$. Prove that

$$\int_0^1 f \cdot \int_0^1 g \geq 1.$$

Problem R-3B. Recall that for A and B nonempty subsets of $\mathbb{R}$, we define

$$d(A,B) := \inf\{d(x,B) : x \in A\},$$

where

$$d(x,B) := \inf\{|x-y| : y \in B\}.$$

Recall, too, that the outer measure m^* is countably subadditive. For arbitrary sets $A, B \subseteq \mathbb{R}$, prove that

$$d(A,B) > 0 \Longrightarrow m^*(A \cup B) = m^*(A) + m^*(B).$$

Problem R-3 cont.

(10 points) **(R-4)**

Choose either Problem R-4A or Problem R-4B below and solve it.

Problem R-4A. On a closed, bounded nondegenerate interval $[a, b]$, consider a sequence $\{f_n\}_{n=1}^{\infty}$ of increasing, absolutely continuous functions on $[a, b]$. Assume:
(a) $f_n(a) = 0$ for all $n \geq 1$;
(b) $f_n' \leq f_{n+1}'$ a.e. on $[a, b]$, for all $n \geq 1$;
(c) the sequence $\{f_n(b)\}_{n=1}^{\infty}$ is bounded.
Prove:

(i) there exists $f : [a, b] \to \mathbb{R}$ such that $f_n \to f$ **pointwise** on $[a, b]$ (as $n \to \infty$);

(ii) f is **absolutely continuous** on $[a, b]$; and

(iii) f_n converges to f **uniformly** on $[a, b]$ (as $n \to \infty$).

Problem R-4B. Let $f_n : [0, 1] \to \mathbb{R}$ be an infinite sequence of functions with

$$\int_0^1 f_n^2 \, dm \leq 1$$

for all $n \geq 1$. Assume that for any function $f : [0, 1] \to \mathbb{R}$ with $\int_0^1 f^2 \, dm < \infty$ we have

$$\lim_{n \to \infty} \int_0^1 f_n \cdot f \, dm = 0.$$

Show that there is an **infinite subsequence** $\{f_{n_k}\}_{k=1}^{\infty}$ of $\{f_n\}_{n=1}^{\infty}$ such that

$$\lim_{k \to \infty} \frac{1}{k^2} \int_0^1 (f_{n_1} + f_{n_2} + \cdots + f_{n_k})^2 \, dm = 0.$$

Problem R-4 cont.

(10 points) **(R-5)**

Choose either Problem R-5A or Problem R-5B below and solve it.

Problem R-5A. A subset A of $\mathbb{R}$ is called **nowhere dense in** $\mathbb{R}$ provided that every open set $G \subseteq \mathbb{R}$ has an open subset U that is disjoint from A, that is, $A \cap U = \emptyset$.

Prove that the Cantor set $C \subseteq [0, 1]$ is nowhere dense in $\mathbb{R}$.

Problem R-5B. Let (X, d) be a metric space. For any nonempty set $E \subseteq X$, define

$$d_E(x) := \inf\{d(x, y) : y \in E\}.$$

Prove that d_E is a **uniformly continuous** function on X.

Problem R-5 cont.

PART II: Complex Analysis

(10 points) **(C-1)**
Choose either Problem C-1A or Problem C-1B below and solve it.

Problem C-1A. Let $\mathbb{D}$, $\overline{\mathbb{D}}$, and $\mathbb{T}$ denote the open unit disc, the closure of the open unit disc, and the unit circle, respectively. Let $f : \mathbb{T} \to \mathbb{C}$ be a continuous function, and define $\hat{f} : \overline{\mathbb{D}} \to \mathbb{C}$ by

$$\hat{f}(z) := f(z) \quad (z \in \mathbb{T})$$

and

$$\hat{f}(re^{i\theta}) := \frac{1}{2\pi} \int_{-\pi}^{\pi} f\left(e^{it}\right) P_r(\theta - t)\, dt \quad (r < 1),$$

where P_r denotes the Poisson kernel.

Given $r < 1$, define $\hat{f}_r : \mathbb{T} \to \mathbb{C}$ by

$$\hat{f}_r(z) := \hat{f}(rz) \quad (z \in \mathbb{T}).$$

Show that for each $r < 1$ there is a sequence $\{p_n(z, \bar{z})\}_{n=1}^{\infty}$ of **polynomials** in z and $\bar{z}$ such that $p_n(z, \bar{z}) \to \tilde{f}_r(z)$ **uniformly** for z in $\mathbb{T}$ (as $n \to \infty$). (Hint: Use Definition 2.1 in Prep Sheet.)

Problem C-1B. Determine if each of the given statements is true or false. If true, provide a proof; if false, provide a counterexample or show in some fashion why the statement is false. In either case, feel free to use results from the Prep Sheet.

Let $\mathbb{D}$ be the open unit disc in $\mathbb{C}$.

(i) (True or False?) There exists an analytic function $f : \mathbb{D} \to \mathbb{D}$ such that

$$f\left(\frac{1}{2}\right) = \frac{2}{3} \quad \text{and} \quad f'\left(\frac{1}{2}\right) = \frac{7}{9}.$$

(ii) Let $f : \mathbb{D} \to \mathbb{D}$ be an analytic function. Assume that f has two distinct fixed points in $\mathbb{D}$.
(True or False?) f is the identity function; that is, $f(z) = z$ for every $z \in \mathbb{D}$.

Problem C-1 cont.

(10 points) **(C-2)**
Choose either Problem C-2A or Problem C-2B below and solve it.

Problem C-2A. **Use complex analysis methods** to show that

$$\int_{-\infty}^{\infty} \frac{\cos(x)}{(x^2+4)(x^2+1)}\,dx = \frac{\pi}{3}\cdot\left(\frac{1}{e}-\frac{1}{2e^2}\right).$$

(Hint: Since $\sin(x)$ is an odd function, it readily follows that

$$\int_{-\infty}^{\infty} \frac{\cos(x)}{(x^2+4)(x^2+1)}\,dx = \int_{-\infty}^{\infty} \frac{e^{ix}}{(x^2+4)(x^2+1)}\,dx.)$$

Problem C-2B. Consider the horizontal strip $\Omega := \{z \in \mathbb{C} : -2 < Im(z) < 2\}$. Show that for any $f : \Omega \to \mathbb{D}$ we have

$$2\pi|f(0)|^2 \le \int_{-\infty}^{\infty} |f(x+i)|^2\,dx + \int_{-\infty}^{\infty} |f(x-i)|^2\,dx.$$

Problem C-2 cont.

(10 points) **(C-3)**

Choose either Problem C-3A or Problem C-3B below and solve it.

Problem C-3A. Determine if the given statement is true or false. If true, provide a proof; if false, provide a counterexample or show in some fashion why the statement is false. In either case, feel free to use results from the Prep Sheet.

Let $u, v \in \mathbb{C}$ and assume that $\bar{u}v \neq 1$.
(True or False?) The following identity holds:

$$1 - \frac{(1 - |u|^2)(1 - |v|^2)}{|1 - \bar{u}v|^2} = \frac{|u - v|^2}{|1 - \bar{u}v|^2}.$$

Problem C-3B. Consider the closed set F in $\mathbb{C}$ defined (in polar form) as follows:

$$F := \{z \equiv (r, \theta) \in \mathbb{C} : \theta \in (0, +\infty) \text{ and } \theta \leq r \leq 2\theta\}.$$

Prove that the region $\Omega := \mathbb{C} \setminus F$ is conformally equivalent to the open unit disc $\mathbb{D}$ in $\mathbb{C}$; that is, there exists a bijective map $\Phi : \Omega \to \mathbb{D}$ such that both Φ and Φ^{-1} are analytic.

Problem C-3 cont.

(10 points) **(C-4)**

Choose either Problem C-4A or Problem C-4B below and solve it.

Problem C-4A. Prove that three out of the four zeros of the equation

$$z^4 - 6z + 3 = 0$$

lie in the annulus

$$\operatorname{ann}(0; 1, 2) := \{z \in \mathbb{C} : 1 < |z| < 2\}.$$

(Hint: Apply Rouché's Theorem, twice.)

Problem C-4B. Suppose that f is holomorphic on the open unit disc $\mathbb{D}$, and continuous on $\overline{\mathbb{D}}$. Assume that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for all $|z| < 1$, and that f admits **exactly** 2025 roots (counted with multiplicities) in $\mathbb{D}$. Show that

$$\inf_{|z|=1} |f(z)| \leq |a_0| + |a_1| + \cdots + |a_{2025}|.$$

Problem C-4 cont.

(10 points) **(C-5)**

Choose either Problem C-5A or Problem C-5B below and solve it.

Problem C-5A. Let $\mathbb{D}$ be the open unit disc in $\mathbb{C}$. Find a concrete description of a Möbius transformation $T(z) = \dfrac{az+b}{cz+d}$ such that

$$T(\mathbb{D}) = \mathbb{D} \qquad \text{and} \qquad T(\frac{i}{2}) = -\frac{i}{2}.$$

Problem C-5B. Let $C(\mathbb{T})$ denote the algebra of continuous functions on the unit circle $\mathbb{T}$, and let

$$A(\mathbb{D}) := \{f \in C(\mathbb{T}) : f \text{ has a continuous extension } F \text{ to } \overline{\mathbb{D}} \text{ such that } F \text{ is analytic in } \mathbb{D}\}.$$

For a function $f \in C(\mathbb{T})$, the Cauchy transform is defined by

$$C(f)(z) := \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{f(w)}{w-z}\, dw \qquad (z \in \mathbb{D}).$$

Let $f \in A(\mathbb{D})$. Prove the following properties of the Cauchy transform:

(i) $C(\bar{f})(z) = \overline{f(0)}$ (for all $z \in \mathbb{D}$).

(ii) $C(\bar{f}) \in A(\mathbb{D})$.

Problem C-5 cont.

Additional Space

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Additional Space

End of Exam