Department of Mathematics University of Iowa Iowa City, IA 52242

118 MLH 9:00 am--12:00 pm, Thursday, August 21, 2025

Ph.D. Qualifying Exam and M.S. Comp. Exam on Numerical Analysis

Note: There are two parts. **In Part I, answer 4 out of 5 questions. In Part II, answer 4 out of 5 questions.** *For each part, if you do all 5 questions, specify which 4 questions are to be graded.* You need 80% for a Ph.D. pass, and 70% for an M.S. pass. Always show your calculations and justify your answers.

Part I

1. Consider a naive implementation of

$$\tanh(x) = \frac{\exp(x) - \exp(-x)}{\exp(x) + \exp(-x)}.$$

   That is, the code directly implements this formula in Julia or Python or Matlab. If the input is $x = \pm 1000$ then the output is NaN instead of approximately ± 1. What is the underlying cause of the problem? Is it catastrophic cancellation, overflow, underflow, or something else?

   From your diagnosis, re-write the code so that it uses equivalent formulas, but avoids returning NaN. You can use if/else statements where appropriate.

2. Carry out two steps of Newton's method to solve $x\,\sin(x) = 2$ starting with $x_0 = 2$.

   Characterize the rate of convergence of iterates produced by Newton's method to a solution, assuming that the iterates converge. Give conditions that ensure this rate of convergence.

3. Explain how divided differences can be used to represent the polynomial interpolant of degree $\leq n$ from the data points $(x_i, f(x_i))$, $i = 0, 1, 2, \ldots, n$. Give formulas so that it can be easily implemented.

   If p_n is the interpolant of degree $\leq n$ of $(x_i, f(x_i))$, $i = 0, 1, 2, \ldots, n$, give the formula for the error $f(x) - p_n(x)$ using $(n+1)$st derivatives of f. Explain how the use of Chebyshev points is good at giving close-to-minimax errors in the polynomial interpolant.

4. It is well known that polynomial interpolation with equally spaced points can be unreliable. This phenomena can be explained through Lebesgue constants.

Let $P\colon C[a,b] \to C[a,b]$ be the interpolation operator $Pf = p$ where p is the interpolant of f according to the given interpolation scheme. Assume the representation

$$p(x) = \sum_{i=1}^{n} f(x_i)\, \ell_i(x),$$

where $\ell_i(x_j) = 1$ if $i = j$ and zero if $i \neq j$. In this representation, determine the operator norm $L := \|P\|$ in terms of the ℓ_i functions. Here L is the Lebesgue constant of the interpolation scheme. See end of Part I for more information about the norms used.

Show that $P^2 = P$. From this show that $\|P\| \geq 1$ or $P = 0$. For polynomial interpolation with $n+1$ equally spaced points, $L \approx 2^{n+1}/(e\, n\, \ln n)$, while for not-a-knot equally spaced spline interpolation $L \leq 3$. How can this explain the lack of robustness of equally spaced polynomial interpolation with many points, while spline interpolation is much more robust?

5. Suppose that $f\colon [a,b] \to \mathbb{R}$ is continuous. The Gaussian quadrature method on n points

$$\int_a^b w(x)\, f(x)\, dx \approx \sum_{i=1}^{n} w_i\, f(x_i),$$

with $w(\cdot)$ a positive integrable weight function, is exact for all polynomials of degree $\leq 2n+1$. Also, all weights $w_i > 0$.

Show that

$$\left| \int_a^b w(x)\, f(x)\, dx - \sum_{i=1}^{n} w_i\, f(x_i) \right| \leq 2 \int_a^b w(x)\, dx\, \|f - q\|_\infty,$$

where q is the minimax approximation to f on $[a,b]$ of degree $\leq 2n+1$.

Explain why this means that the error in the Gaussian quadrature formula must go to zero as $n \to \infty$.

Note that the operator norm of a linear operator $A\colon V \to W$ is given by

$$\|A\| = \sup_{v \neq 0} \frac{\|Av\|_W}{\|v\|_V} = \sup_{v:\|v\|_V \leq 1} \|Av\|_W.$$

The norm used for $C[a,b]$ is

$$\|g\|_{C[a,b]} = \max_{x \in [a,b]} |g(x)|\,.$$

Part II

1. Find the LU factorization of the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 3 & 5 & 2 \\ 0 & 4 & 3 \end{pmatrix}.$$

2. Analyze the convergence of the residual correction scheme

$$r^{(m)} = b - Ax^{(m)}, \qquad x^{(m+1)} = x^{(m)} + Cr^{(m)}, \qquad m \geq 0$$

with

$$A = \begin{pmatrix} 2 & \epsilon \\ -10\,\epsilon & 5 \end{pmatrix}, \qquad C = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.2 \end{pmatrix},$$

where ϵ is a real number. Find the interval of values of ϵ for which the iteration will converge to the solution of $Ax = b$ for all choices of the initial value $x^{(0)}$.

3. For the linear system

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix},$$

provide a complete description of the GS method, and determine if the Gauss-Seidel method converges for all choices of the initial value.

4. Give the Euler method and the backward Euler method for solving the initial-value problem

$$\begin{cases} y' = \cos(x^2 y), \\ y(0) = 1 \end{cases}$$

for $x \in [0, 1]$.

5. Consider the initial value problem

$$y' = f(t, y), \quad y(0) = y_0.$$

Assume that the exact solution and the right hand side $f(t, y)$ are sufficiently smooth. With a parameter $\theta \in [0, 1]$ and a constant step-size h, we use the one-step method

$$u_{n+1} = u_n + h\left[\theta\, f(t_n, u_n) + (1 - \theta)\, f(t_{n+1}, u_{n+1})\right], \; n \geq 0, \quad u_0 = y_0.$$

It is known that for h small enough, this method is well-defined.
(a) Determine the order of the method.
(b) Determine the range of θ for which the method is A-stable.