Ph.D. Qualifying Exam in Topology

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Instructions.

- Do eight problems: four from part A and four from part B.
- This is a closed book examination: you should have no books, technology or paper of your own. Paper will be provided by the test center.
- Please do your work on the paper provided according to the format outlined below.
  - On each page of your solutions
    - Write your name
    - Write the page number
    - Indicate which problem is being addressed
  - When you start a new problem, start a new page
  - Only write on one side of the paper
  - Make a cover page and indicate which eight problems you want graded.
- Always justify your answers unless explicitly instructed otherwise.
- You may use theorems if the problem is not a step in proving that theorem. You must to state any theorems that you use clearly and carefully.

Part A - Algebraic Topology

In the problems below, the symbols for the disk D^n, the sphere S^n and the simplex Δ^n can be understood to mean the subspaces below

$$D^n := \{v \in \mathbb{R}^n : |v| \leq 1\} \qquad S^n := \{v \in \mathbb{R}^{n+1} : |v| = 1\}$$
$$\Delta^n := \{(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} : \sum_{i=0}^{n} x_i = 1\}$$

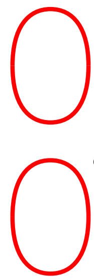
1. Let $T := S^1 \times S^1$ be the torus, $\gamma(t) := (e^{2\pi i t}, 1)$ the loop in T and similarly $\eta(t) := (1, e^{2\pi i t})$. Let $S := (T \sqcup T)/\sim$ be the disjoint union of two copies of the torus with the images of γ and η identified using the equivalence relation
$$\gamma(t) \sim \eta(t)$$
for each $t \in [0, 1]$. Compute $\pi_1(S, \eta(0))$.

2. Let W_n be the product of spheres
$$W_n := S^0 \times S^1 \times S^2 \times \cdots \times S^n$$
Compute the homology groups $H_m(W_7)$ for all $m \in \mathbb{Z}$.

3. Let $\gamma : S^1 \to T$ be a curve in a torus $T := S^1 \times S^1$ and let
$$\Gamma := \gamma_*(1) \subset \pi_1(T, \gamma(1))$$
be the subgroup generated by γ. Construct the cover $p_\Gamma : \tilde{T} \to T$ associated to the subgroup Γ. Construct the lift map $\tilde{\gamma} : S^1 \to \tilde{T}$ and prove that $p_\Gamma \circ \tilde{\gamma} = \gamma$.

4. Let $\Delta^n_k \subset \Delta^n$ be the subspace consisting of all k-dimensional faces. Compute the homology groups of the quotient space Δ^5/Δ^5_3.

5. Let U and H be the unlink and the Hopf link, pictured on the left-hand and right-hand sides below. Each picture describes the embedding of two circles $S^1 \sqcup S^1$ into the 3-ball D^3. Prove that the spaces $D^3 \backslash U$ and $D^3 \backslash H$ are not homeomorphic.



U

H

6. Let $p : \tilde{X} \to X$ be a covering map. Prove that the group homomorphism $p_* : \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, x_0)$ is injective for any points $x_0 \in X$ and $\tilde{x}_0 \in p^{-1}(x_0)$.

Part B - Manifolds and vector bundles

1. Let $f\colon M \longrightarrow N$ be a continuos map between smooth manifolds. Show that f is a smooth map if and only if for every smooth function $h\colon N \longrightarrow \mathbb{R}$, the composition function $h \circ f\colon M \longrightarrow \mathbb{R}$ is also smooth.

2. Prove that $\mathrm{O}(n) = \{A \in M_{n\times n}(\mathbb{R}) : A^T A = \mathrm{Id}\}$ is a manifold and find its dimension.

3. Suppose $\gamma : S^1 \to \mathbb{R}^2$ is a smooth embedding. What are the possible values of the integral $\int_\gamma \eta$ if

$$\eta = \frac{-y}{x^2+y^2}\, dx + \frac{x}{x^2+y^2}\, dy.$$

4. Let

$$X_1 = \sum_{i=1}^{n} x_i \partial_{y_i} \quad \text{and} \quad X_2 = \sum_{i=1}^{n} y_i \partial_{x_i}$$

be vector fields on $\mathbb{R}^{2n}$ with coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$. Compute $[X_1, X_2]$.

5. Show that the map $\varphi\colon \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$\varphi(x, y, z) = (2y, -x, -xy + z)$$

is a diffeomorphism. Let $X = x\partial_x + y\partial_y$ be a vector field on $\mathbb{R}^3$. If the pushforward $d\varphi(X)$ is expressed in coordinates as

$$d\varphi(X) = a\partial_x + b\partial_y + c\partial_z,$$

find the coefficient functions a, b, and c.

6. Calculate the de Rham cohomology groups of S^2 (You may use Mayer-Vietoris)