

Ph.D. qual. exam. and M.S. comp. exam. on ODEs.
Monday August 18, 2025.

Answer 5 out of 9 questions (only the best 5 will count). Show your calculations and justify your answers.

1. Give an expression for $\det(R(t, t_0))$ (the determinant of the matrix $R(t, t_0)$) where the matrix $R(t, t_0)$ is the transition matrix of the following system of nonautonomous linear ODEs

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \sin^3(t) \cos(t)x_1 + \sin^2(t) \cos^2(t)x_2 \\ \sin^2(t) \cos^2(t)x_1 + \sin(t) \cos^3(t)x_2 \end{bmatrix}.$$

2. Consider a nonautonomous linear homogeneous system of ODEs $\dot{x} = A(t)x$. When $A(t)$ is skew-symmetric ($A(t)^T = -A(t)$), show that the transition matrix $R(t, t_0)$ of $\dot{x} = A(t)x$ must be orthogonal, i.e., $R(t, t_0)^T R(t, t_0) = I$.
3. The origin $x^* = (0, 0, 0)$ is a fixed point of the autonomous system of differential equations

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -2x_2 + x_2x_3 - x_1^3 \\ x_1 - x_1x_3 - x_2^3 \\ x_1x_2 - x_3^3 \end{bmatrix}.$$

- (a) Show that $x^* = (0, 0, 0)$ is a nonhyperbolic fixed point.
 - (b) Is the function $V(x_1, x_2, x_3) := x_1^2 + 2x_2^2 + x_3^3$ a (strict) Lyapunov function at the origin $x^* = (0, 0, 0)$ for this system of differential equations?
 - (c) Is the fixed point $x^* = (0, 0, 0)$ a sink (i.e., asymptotically Lyapunov stable)?
4. Consider the system of ODEs given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 - 9x_1(x_1^2 + x_2^2) \\ x_1 - x_2 - 9x_2(x_1^2 + x_2^2) \end{bmatrix}.$$

Does this system have a periodic orbit? Note: a fixed point is not considered as being a periodic solution.

5. We consider the *damped Duffing equations*

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = x_1 - x_1^3 - \delta x_2,$$

where the constant δ satisfies $\delta > 0$. Does this system have a periodic orbit or not?

6. Consider the system of ODEs given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -3x_2 - 2x_1(1 - 4x_1^2 - 4x_2^2) \\ 3x_1 - 2x_2(1 - 4x_1^2 - 4x_2^2) \end{bmatrix}$$

which has periodic solution

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \cos(3t) \\ \frac{1}{2} \sin(3t) \end{bmatrix},$$

and thus periodic orbit

$$\Gamma = \left\{ (x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 = \frac{1}{4} \right\}.$$

Is this periodic orbit Γ orbitally stable?

7. Find a function $y \in C^1([0, \ln(3)], \mathbb{R})$ satisfying $y(0) = 1, y(\ln(3)) = -1$ such that the first variation of

$$\mathcal{A}[y] := \int_0^{\ln(3)} \left(\frac{y'}{2+y} \right)^2 dx$$

satisfies $\delta\mathcal{A}[y] \equiv 0$, i.e., $\delta\mathcal{A}[y][h] = 0 \forall h \in C_0^1([0, \ln(3)], \mathbb{R})$.

8. For the following scalar ODE

$$\dot{x} = \mu + x - e^x.$$

find all the fixed points as a function of the parameter $\mu \in \mathbb{R}$, determine their Lyapunov-stability, sketch a bifurcation diagram, and find the bifurcation value(s) of the parameter $\mu \in \mathbb{R}$. Determine the bifurcation(s) as being of the type *fold (saddle-node)*, *transcritical*, or *supercritical/subcritical pitchfork*.

9. Consider the system of ODEs $\dot{x} = f(t, x)$ and the following explicit Runge-Kutta method

$$\begin{aligned} X_1 &= x_0 \\ X_2 &= x_0 + h \frac{1}{3} f(t_0, X_1) \\ X_3 &= x_0 + h (-f(t_0, X_1) + 2f(t_0 + h/3, X_2)) \\ x_1 &= x_0 + h \left(\frac{3}{4} f(t_0 + h/3, X_2) + \frac{1}{4} f(t_0 + h, X_3) \right) \end{aligned}$$

- (a) What is the local order of this method?
 (b) Give the value after 10 steps of this Runge-Kutta method applied with constant stepsize $h = 0.2$ to the initial value problem

$$\dot{x} = -10x, \quad x(0) = 1.$$

- (c) Is this Runge-Kutta method applied to $\dot{x} = -10x$ linearly stable for the constant stepsize $h = 0.3$?