Name:__________________________________

Ph.D. Qualifying Exam in PDE

August 2025

Note to students: This booklet contains four problems with equal weights on six sheets of paper including this cover and the extra blank sheet on the back.

You can use the back of the sheets and the extra blank page at the back.

Notes to Students and Proctor: Students are only allowed to bring pens, pencils and erasers. Backpacks and any electronic devices are strictly prohibited.

Any unauthorized item or wrong behavior during the exam will be handled according to the guidelines of the department and that of the Graduate College.

Additinal Note to Proctor: This is PDE part of the Differential Equation qualifying exam, which consists of ODE and PDE parts. Please print the exam one-sided on six pages of paper, extra blank papers can be provided upon requests.

1. a. Define Sobolev Space $W^{1,p}(B_1)$ where $1 \le p \le \infty$ and B_1 is the unit ball in the $n-$dimensional Euclidean space; and define the weak derivatives for functions in $W^{1,p}(B_1)$.

   b. State Sobolev or Morrey embedding theorems for $W^{1,3}(B_1)$ for dimension $n = 1$ and $n = 3$ respectively; and write down these embedding theorems as inequalities for functions in $W^{1,3}(B_1)$.

2. a. Suppose u is a smooth harmonic function defined in the domain $B_1^C = \{X = (x_1, x_2, x_3) : x_1^2 + x_2^2 + x_3^2 > 1\}$, the exterior of the unit ball in three dimensional space and suppose that $u = 0$ on ∂B_1 and that $\lim_{X \to \infty} u = 1$. Prove there is a unique such harmonic function.

   b. Is the above conclusion still true in 2 dimensional space? Prove your statement.

3. Let $\Omega = B_2 \backslash B_1$ in $\mathbf{R}^n$. For $f \in L^2(\Omega)$ and $g \in L^2(\partial B_1)$, define and prove there is a unique weak solution to boundary value problem

$$\begin{cases} -\Delta u &= f, \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} + 2u &= g, \quad \text{on } \partial B_1, \\ u &= 2, \quad \text{on } \partial B_2. \end{cases}$$

4. Consider the heat equation on the positive x-axis

$$\begin{cases} u_t - u_{xx} &= 0, \quad x \in \mathbb{R}^+, \quad 0 < t < \infty, \\ u(x,0) &= x^2, \\ u(0,t) &= 0. \end{cases}$$

   Show there is a classical solution and prove its uniqueness.