Ph.D. Qualifying Exam and M.S. Comprehensive Exam in Algebra

Professors Ryan Kinser and Yangbo Ye

January 11, 2024, 9:00am - 12:00pm, 113 MLH

Instructions:

- Do EXACTLY TWO problems from EACH of the four sections.
- Please start a new page for every new problem and put your name on each sheet.
- Please ONLY WRITE on ONE SIDE of each sheet.
- Justify your answers and show your work.
- Please write legibly.
- In answering any part of a question, you may assume the results in previous parts of the SAME question, even if you have not solved them.
- Please turn in the exam questions with your solutions.

Notations:

We adopt standard notations. Namely:

- We write $\mathbb{C}, \mathbb{R}$ and $\mathbb{Q}$ to denote the field of complex numbers, real numbers and rational numbers, respectively; we write $\mathbb{Z}$ to denote the ring of rational integers; when p is a prime number, we write $\mathbb{F}_p$ to denote the finite field with p elements.
- Throughout this exam, R denotes a ring with identity 1; R is called an integral domain if $1 \neq 0$ and R is commutative with no zero divisors.
- All R-modules are assumed to be unital left R-modules.

1 Groups

1. (15 points) Let G be a finite abelian group of order n. Prove that G is cyclic if and only if for each positive integer d dividing n, there exists a unique subgroup of order d.
   Note that the question is about uniqueness of subgroups, not about uniqueness of subgroups under isomorphisms.
   (You need to prove this using basic techniques learned in MATH:5000. You may use Lagrange's Theorem and/or isomorphism theorems, but you cannot use a theorem which trivializes the question. You may not use any kind of theorem of the form "A group of order n has a subgroup of order d." In particular, you cannot use Sylow theorems.)

2. (a) (10 points) Construct a group of order 57 which is *not* cyclic.
   (You need to prove this using basic techniques learned in MATH:5000. You may not quote any kind of theorem of the form "All groups of order .... are of the form ....")

   (b) (5 points) Let G be a group of order 57 which is not cyclic. Determine (with careful proof) the number of elements of G of each possible order.

3. Let G be any group.
   (a) (5 points) Define the *commutator subgroup* $[G, G] \trianglelefteq G$ and prove it is a characteristic subgroup of G (you may assume it is a subgroup; only prove it is characteristic).

   (b) (5 points) Let $N \trianglelefteq G$ a normal subgroup and assume that $N \cap [G, G] = 1$, the trivial subgroup of G. Prove that $N \leq Z(G)$.

   (c) (5 points) Determine (with proof) the commutator subgroup of the symmetric group S_8.

2 Rings

1. Let R be a commutative ring with $1 \neq 0$, and let I be a proper ideal of R.

   (a) (5 points) Prove that the set S of ideals containing I and $\neq R$ is inductively ordered by ascending inclusion. Note that S is non-empty because $I \in S$.

   (b) (5 points) By Zorn's lemma it is known that there exists a maximal element M in S. Prove that this M is a maximal ideal containing I.

   (c) (5 points) Determine whether a maximal ideal containing I is always unique. If you claim it is always unique, prove it. If it is not always unique, prove a counterexample.

2. Let D be an integral domain.
   (a) (5 points) For $a, b \in D$, define a *greatest common divisor* of a and b.
   (Note that you need to define any terms you use which are not part of the definition of a ring, such as "divisor" for example.)

   (b) (5 points) Prove that if $(a) + (b) = (d)$, then d is a greatest common divisor of a and b.

   (c) (5 points) Prove that any two greatest common divisors of a and b are unit multiples of one another.

3. Let $\mathbb{Z}[\frac{1}{2}] \subset \mathbb{Q}$ be the subring generated by $\mathbb{Z}$ and $\frac{1}{2}$; explicitly, its elements are exactly those fractions of the form $\frac{a}{2^n}$ where $a \in \mathbb{Z}$ and $n \geq 0$.

   (a) (6 points) Prove that every ideal of $\mathbb{Z}[\frac{1}{2}]$ is generated by a single integer.

   (b) (9 points) By constructing an appropriate isomorphism, use part (a) to determine all ideals in the polynomial ring $\mathbb{Z}[x]$ which contain the ideal $(2x - 1)$, where x is a variable.

   You should solve this problem using basic properties of rings and ideals learned in MATH:5000–5010. You should not quote any theorem which trivializes the solution.

3 Linear Algebra and Module Theory

1. Let R be a ring with $1 \neq 0$ and let E be an R-module (i.e. a unital left R-module).

   (a) (5 points) Denote by $R' = \mathrm{End}_R(E)$ the ring of R-endomorphisms. Prove that E is also an R'-module (i.e. a unital left R'-module).

   Denote $R_0 = R$. For $n \geq 1$, one may use Part (a) to inductively define $R_n = \mathrm{End}_{R_{n-1}}(E)$ to be the ring of R_{n-1}-endomorphisms of E for E as an R_{n-1}-module and then regard E as an R_n-module. This way one gets an infinite sequence of rings $R_0, R_1, R_2, \ldots\ldots$ and an infinite sequence of modules:

   $$E \,\text{as an}\, R_0\text{−module}, E \,\text{as an}\, R_1\text{−module}, E \,\text{as an}\, R_2\text{−module}, \ldots\ldots.$$

   (b) (10 points) Consider the special case of $R = R_0 = k$ being a field and E being a finite dimensional vector space over k. Determine the ring sequence $R_0, R_1, R_2, \ldots\ldots$ and the module sequence:

   $$E \,\text{as an}\, R_0\text{−module}, E \,\text{as an}\, R_1\text{−module}, E \,\text{as an}\, R_2\text{−module}, \ldots\ldots.$$

2. Consider the matrix $A = \begin{bmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}$, say over $\mathbb{C}$.

   (a) (3 points) Determine the eigenvalues of A, and for each such eigenvalue λ, a basis of the corresponding eigenspace E_λ.

   (b) (4 points) Determine whether A is diagonalizable or not: if so, give an explicit matrix P such that $P^{-1}AP$ is diagonal, and if not, give a proof.

   (c) (4 points) State the Jordan canonical form of A with a brief explanation why your answer is correct.

   (d) (4 points) State the elementary divisor decomposition of the corresponding $\mathbb{C}[x]$-module, with a brief explanation.

3. Let V be the vectors space of $n \times n$ matrices over a field F. Let U be the subspace of symmetric matrices (i.e. those matrices A satisfying $A = A^T$) and W be the subspace of skew-symmetric matrices (i.e. those matrices A satisfying $A = -A^T$).
   (a) (8 points) If $F = \mathbb{R}$, prove that $V = U \oplus W$.

   (b) (5 points) Give an example, with proof, of a field F such that $V \neq U \oplus W$.

   (c) (2 points) State a necessary and sufficient condition on F for $V = U \oplus W$ to hold. *(You don't need a proof, it is basically obvious from the solutions of the earlier two parts.)*

4 Field Theory and Galois Theory

1. (5 points each) Consider the field $F = \mathbb{Q}(\sqrt[12]{3})$, where $\sqrt[12]{3}$ is the positive 12th root of 3.

   (a) Find the smallest normal extension K of $\mathbb{Q}$ containing F.

   (b) Find the smallest subfield E of F such that F is normal over E.

   (c) Find the Galois groups $\mathrm{Gal}(K/\mathbb{Q})$, $\mathrm{Gal}(K/E)$, $\mathrm{Gal}(K/F)$, and $\mathrm{Gal}(F/E)$.

2. (15 points) Let F be a field, $f(x) \in F[x]$ an irreducible polynomial, and α a root of f in some extension. Assume f has an odd degree term with nonzero coefficient. Prove that $F(\alpha) = F(\alpha^2)$.
   Hint: by separating even and odd degree terms, an arbitrary polynomial can be written in the form $g(x^2) + xh(x^2)$ *for some* $g, h \in F[x]$.

3. Let L be a Galois extension of k and let $k \subseteq E \subseteq L$ and $k \subseteq F \subseteq L$ be intermediate fields.

   (a) (5 points) Prove that K is Galois over each of E, F, and EF.

   (a) (10 points) Prove that $\mathrm{Gal}(K/EF) = \mathrm{Gal}(K/E) \cap \mathrm{Gal}(K/F)$.