Math@Iowa Analysis Qualifying Exam
Friday, January 12, 2024
9:00 am - 12:00 noon in Room 113 MLH

Examination Committee: Professors Ionuţ Chifan and Raúl E. Curto

Your Name:

Instructions

1. This is a closed-book exam; no books or notes are allowed. This exam includes a Prep Sheet (attached), which you may use as a reference.
2. You are not allowed to use any internet resources; use of calculators is not allowed.
3. This Qualifying Exam consists of 10 questions, broken down into two parts of 5 questions each, for a maximum score of 70 points. Please submit a total of seven questions, with at least three chosen from each part. Please summarize your choices in the table below, by placing a ✓ inside the box located next to each choice.
5. Please write your answers within the space provided for each question. If at all possible, please write your answers in ink (blue, black or red). However, if writing math using a pencil is what you always do, by all means use a pencil for this exam too.
6. For each solution you provide, make sure to include all the details, and state correctly all the known results you use. If appropriate, feel free to refer to the attached Prep Sheet.
7. A score in the range **63 − 70** guarantees a **Ph.D. Pass**; a score in the range **56 − 62** guarantees an **M.S. Conditional Pass**.
8. **Before you start, please write your name on the space provided in this page. When you finish, please turn in this exam booklet.**

☐ R-1A	☐ R-1B		☐ C-1	
☐ R-2			☐ C-2A	☐ C-2B
☐ R-3A	☐ R-3B		☐ C-3	
☐ R-4			☐ C-4A	☐ C-4B
☐ R-5A	☐ R-5B		☐ C-5A	☐ C-5B

PART I: Real Analysis

(10 points) **(R-1)**

Choose either Problem R-1A or Problem R-1B below and solve it.

Problem R-1A. Let F be a generalized Cantor set of positive measure; that is, F is built using the middle-third algorithm in the construction of the classical Cantor set, but using $\frac{\alpha}{3^n}$ as the length of a removed third instead of $\frac{1}{3^n}$, with $0 < \alpha < 1$.

Let $E := [0,1] \setminus F$, and define

$$f(x) := \int_{[0,x]} \chi_E \qquad (0 \leq x \leq 1).$$

(a) Prove that f is absolutely continuous on $[0,1]$.

(b) Prove that f is strictly increasing on $[0,1]$.

(c) Prove that $f' = 0$ on a set of positive measure.

Problem R-1B. Let $E := [0,1]$, $g \in L^2(E)$, and define $f : E \to \mathbb{R}$ by

$$f(x) := \int_{(0,x)} g \qquad (x \in E).$$

(a) Prove that f is well defined; that is, for every $g \in L^2(E)$ one also has $g \in L^1(E)$, and therefore $f(x)$ is a real number.

(b) Prove that f may fail to be Lipschitz on E; that is, find $g \in L^2(E)$ such that the associated f is not Lipschitz on E.

(c) Prove that, when $g \in L^\infty(E)$, then f is Lipschitz.

(d) For $g \in L^\infty$, find the best Lipschitz constant for f.

Problem R-1 cont.

(10 points) **(R-2)** Let $a < b \in \mathbb{R}$, and let f and g be two absolutely continuous functions on $[a, b]$. Assume that there exist positive constants M_f and M_g such that

$$|f'| \leq M_f$$

almost everywhere on $[a, b]$, and similarly

$$|g'| \leq M_g$$

almost everywhere on $[a, b]$. On the interval $[a, b]$, consider the function $h := \max\{f, g\}$.

Prove that h is Lipschitz on $[a, b]$.

Problem R-2 cont.

(10 points) **(R-3)**

Choose either Problem R-3A or Problem R-3B below and solve it.

Problem R-3A. Let $E := [0,1]$ and let $f \in L^2(E)$ such that $\int_E f = 0$ and $\int_E f^2 = 1$. Prove that

$$m(\{f > t\}) \leq \frac{1}{1+t^2},$$

for all $0 \leq t \leq 1$.

Problem R-3B. Let log denote the natural logarithmic function, and for $0 < x \leq 1$ let $f(x) := \log(\frac{1}{x})$.

(i) Prove that $f \in \bigcap_{1 \leq p < \infty} L^p((0,1])$.

(ii) Prove that $f \notin L^\infty((0,1])$.

(Hint: First prove, using mathematical induction and integration by parts, that

$$\int_{(0,1]} f^k = k!$$

for all positive integers k. Then use this fact to interpolate $\int_{(0,1]} f^p$ for non-integer values of p.)

Problem R-3 cont.

(10 points) **(R-4)** Let f be a uniformly continuous, integrable function on $[-\infty, \infty)$.

(a) Prove that both $R := \lim_{x \to \infty} f(x)$ and $L := \lim_{x \to -\infty} f(x)$ exist, and that $L = R$.

(b) Is the conclusion in (a) still true if f is merely continuous instead of uniformly continuous?

Problem R-4 cont.

(10 points) **(R-5)**

Choose either Problem R-5A or Problem R-5B below and solve it.

Problem R-5A. Determine if the given statement is true or false. If true, provide a proof; if false, provide a counterexample or show in some fashion why the statement is false. In either case, feel free to use results from the Prep Sheet.

A left half-open interval in $\mathbb{R}$ is an interval of the form $(a, b]$ or $(a, +\infty)$, where $-\infty \leq a < b$. Let $\mathcal{F}$ be the collection of left half-open intervals in $\mathbb{R}$ and let $\mathcal{A}(\mathcal{F})$ be the algebra of sets generated by $\mathcal{F}$, that is, the collection of **finite** unions of intervals in $\mathcal{F}$.

(True or False) $\mathbb{R} \setminus \mathbb{Q} \in \mathcal{A}(\mathcal{F})$.

Problem R-5B. Let $f \in L^2([0,1])$. Extend f to all of $\mathbb{R}$ by setting it identically equal to zero outside the interval $[0, 1]$.

(i) Prove that

$$\lim_{x \to 0} \int_{[0,1]} [f(x+t) - f(t)]^2 dt = 0.$$

(ii) Prove that the real-valued function F defined on $\mathbb{R}$ by

$$F(x) := \int_{[0,1]} f(x+t) \cdot f(t) dt \quad (-\infty < x < \infty)$$

is continuous at $x = 0$.

Problem R-5 cont.

PART II: Complex Analysis

(10 points) **(C-1)** **Use complex analysis methods** to calculate

$$\int_0^\infty \frac{x^2}{(x^2+1)(x^2+4)}\, dx.$$

Problem C-1 cont.

(10 points) **(C-2)**

Choose either Problem C-2A or Problem C-2B below and solve it.

In Problems C-2A and C-2B, determine if the given statement is true or false. If true, provide a proof; if false, provide a counterexample or show in some fashion why the statement is false. In either case, feel free to use results from the Prep Sheet.

Problem C-2A. (True or False) The function

$$\frac{z^2(z-1)}{(1-\cos z)\log(1+z)}$$

has a simple pole at $z = 0$.

Problem C-2B. (True or False) Let f be analytic on the open unit disc $\mathbb{D}$, and assume that $|f(z)| < 1$ on $\mathbb{D}$. Assume also that there exist two distinct points a, b in $\mathbb{D}$ which are fixed points for f (i.e., $f(a) = a$ and $f(b) = b$). Then $f(z) = z$ for all $z \in \mathbb{D}$.

Problem C-2 cont.

(10 points) **(C-3)** Consider the polynomial $p(z) := 2z^5 - 6z^2 + z + 1$. Determine the number of zeros of p (counting multiplicities) inside the annulus $\text{ann}(0; 1, 2)$ of radii 1 and 2 centered at the origin.

(10 points) **(C-4)**

Choose either Problem C-4A or Problem C-4B below and solve it.

Problem C-4A. Consider the square-root mapping $w = \sqrt{z}$, defined in the region

$$G := \mathbb{C} \setminus \{z \in \mathbb{R} : z \leq 0\},$$

and let $a, b \in \mathbb{R}$ such that $0 < a < b$. Describe in complete detail the image under this mapping of the horizontal strip

$$S := \{z \in \mathbb{C} : a < \operatorname{Im} z < b\}.$$

Problem C-4B. On the space $H(\mathbb{D})$ of analytic functions on the open unit disc, consider the Hilbert and Cèsaro transforms, defined as

$$H(f)(z) := \int_0^1 \frac{f(t)}{1 - tz}\, dt \quad (f \in H(\mathbb{D}), z \in \mathbb{D})$$

and

$$C(f)(z) := \frac{1}{z} \cdot \int_0^z \frac{f(t)}{1 - t}\, dt \quad (f \in H(\mathbb{D}), 0 \neq z \in \mathbb{D}).$$

(a) Given the sequence of functions $e_n(z) := z^n \quad (z \in \mathbb{D})$, where n is a positive integer, prove that

$$C(e_n) = e_n H(e_n) \quad (\text{for all } n \geq 1).$$

(b) Use the above identity to show that for every $n \geq 1$, the function $C(e_n)$ has a removable singularity at $z = 0$.

Problem C-4 cont.

(10 points) **(C-5)**

Choose either Problem C-5A or Problem C-5B below and solve it.

Problem C-5A.

Let U be a simply connected open set and let $z_1, \dots, z_n$ be points of U.

Let $U^* = U \setminus \{z_1, \dots, z_n\}$ be the set obtained from U by deleting the points $z_1, \dots, z_n$. Let f be analytic on U^*. Let γ_k be a small circle centered at z_k and let

$$a_k = \frac{1}{2\pi i} \int_{\gamma_k} f(\zeta) d\zeta.$$

For $z \in U^*$, let

$$h(z) := f(z) - \sum \frac{a_k}{(z - z_k)}.$$

Prove that there exists an analytic function H on U^* such that $H' = h$.

(Hint: Fix a point $w \in U^*$. Given a path γ in U^* from w to a point $z \in U^*$ define

$$H_\gamma(z) = \int_\gamma h(\zeta) d\zeta.$$

First prove that this function is independent of the path chosen from w to z.)

Problem C-5B. Let

$$f(z) = \frac{\sin z}{z(z+i)(z-2i)} \quad (z \neq 0, z \neq -i, z \neq 2i).$$

Find the Laurent Expansion of f, in powers of z, in the annulus $\operatorname{ann}(0; 1, \infty)$.

Problem C-5 cont.

Additional Space

Additional Space

End of Exam