Numerical Analysis PhD Qualifying Exam and MS Comprehensive Exam

Tuesday, January 9, 2023, 1:00 pm

There are two Parts. **In Part I, answer at least 4 out of 5 questions. In Part II, answer at least 4 out of 5 questions.** Only the best 4 of each Part will count. You need 80% for a PhD pass, and 70% for an MS pass. Always show your calculations and justify your answers.

Part I, answer at least 4 out of 5 questions:

1. Prove that the Lagrange shape functions $\ell_i(x)$ (for $i = 1, \ldots, n$ with $x_j \neq x_k$ for $j \neq k$) satisfy
$$\sum_{i=1}^{n} \ell_i(x) = 1,$$
for all x and for any n. No credit will be given if you only show this to be true for a specific n.

2. Is
$$s(x) := \begin{cases} -4x^2 + 2x^3, & \text{for} \quad x \in [0,1], \\ 2 - 6x + 2x^2, & \text{for} \quad x \in [1,3], \end{cases}$$
a cubic spline on the interval $[0,3]$ for the data points (0,0), (1,-2), (3,2)? To receive credit for this problem, you must state the definition of a cubic spline and show that $s(t)$ either does or does not satisfy this definition.

3. Find the weights w_1, w_2 and nodes x_1, x_2 for the symmetric quadrature formula of form
$$\int_{-1}^{1} f(x)\, dx \approx w_1 f(x_1) + w_2 f(x_2)$$
with highest possible degree of precision (or equivalently order of accuracy).

4. Find the near minimax polynomial approximation of degree at most 2 to the function $f(x) = |x|$ on the interval $[-1, 1]$.

5. Define Newton's method and the secant method to find a zero to a scalar nonlinear equation $g(x) = 0$ with $g : [a, b] \to \mathbb{R}$. Give an advantage of the secant method compared to Newton's method.

Part II, answer at least 4 out of 5 questions:

1. Consider the matrix
$$A := \begin{bmatrix} 2 & 1 \\ -2 & 0 \\ 1 & 1 \end{bmatrix} \in \mathbb{R}^{3\times 2}.$$
Find $x \in \mathbb{R}^2$ minimizing $\|b - Ax\|_2$ for
$$b := \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} \in \mathbb{R}^3.$$

2. Take one step of the Trapzoid Rule Method for ODEs for the equation

$$\dot{x}(t) = \cos(t)x(t),$$

with $x(0) = 1$, starting at $t = 0$ and using step size $h = 0.2$.

3. Derive the "symbol" (stability function) for Heun's Method.

4. Show that if A is real, square, and invertible, then the QR factorization is unique apart from a diagonal scaling by factors of ± 1. That is, if $A = Q_1 R_1 = Q_2 R_2$ with Q_1, Q_2 orthogonal and R_1, R_2 upper triangular, then there is a diagonal matrix D with diagonal entries ± 1 where $Q_2 = Q_1 D$ and $D R_2 = R_1$. In particular, show that if the diagonal elements of R_1 and R_2 are positive, then $Q_1 = Q_2$ and $R_1 = R_2$.

5. An $n \times n$ matrix is strictly diagonally dominant if

$$|a_{ii}| > \sum_{j:j\neq i} |a_{ij}|$$

for all i. Show that strictly diagonally dominant matrices are invertible.