# Ph.D. Qual Exam Diff Eqs
# January 10, 2024
# Part 1: ODE/Nonlinear Dynamics

## Student Name:

Note to Proctor: Please print this document as is (front and back), using the printer option "on both sides of the paper". Thanks!

### Instructions for Part 1:

- Solve 3 out of the 4 problems included on the next two pages.
- If you work on all four problems, please **clearly state** -on the front page, near you name - which 3 problems should be graded.
- In case you attempt all four problems and forget to indicate which three to be graded, the score will be computed based on the results from problems 1-3. Please make sure you clearly state your preference.
- All problems have **equal** points.
- Please **start a new page** for every new problem and write your name on each sheet.
- Please turn in the exam questions with your solutions.
- Please turn in the scratch papers. All scratch papers will be discarded.

**Good luck!**

**Problem 1.** Consider the dynamical system

$$\begin{cases} \dot{x} = x(1 - x^2 - y^2) - y \\ \dot{y} = y(1 - x^2 - y^2) + x \end{cases} \tag{1}$$

a) Show that $\gamma(t) = (\cos t, \sin t)$ is a periodic solution of system (1), then construct $\dot{Y} = A(t)Y$, the linearization of system (1) about $\gamma(t)$.

b) Apply the Floquet theory to the linear system in (a) to determine the Floquet multipliers $\mu_1$, $\mu_2$. Determine if the periodic solution $\gamma(t)$ is **linearly** stable or unstable.

c) Write system (1) in polar coordinates $(r, \theta)$ according to $x = r\cos\theta$, $y = r\sin\theta$. Use the result to determine if the periodic solution $\gamma(t)$ is **nonlinearly** asymptotically stable or unstable.
Hint: Use formulas $\dot{r} = (x\dot{x} + y\dot{y})/r$, $\dot{\theta} = (x\dot{y} - y\dot{x})/r^2$.

d) Draw the phase plane for system (1), i.e. sketch "landmark"-trajectories in the plane and sketch other typical trajectories around those landmarks.

**Problem 2.** Consider the dynamical system associated with the linear ODE $X' = AX$ where $A$ is given by

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 1 & -2 & -2 \\ 1 & 0 & 2 \end{bmatrix} \tag{2}$$

a) Determine the dimension of each subspace $E^u$ (unstable), $E^c$ (center) and $E^s$ (stable).

b) Determine if the system's equilibrium is: (i) asymptotically stable? (ii) spectrally stable? (iii) unstable? (iv) hyperbolic? Justify your answers.

c) Find the projection of the ODE system $X' = AX$ with $A$ defined by (2) on the subspace $E^u$, i.e. find $X_u' = \mathcal{U}X_u$.

d) Find the dynamical system, i.e. **the flow**, $(\mathbf{M}, (\Phi_t)_{t\in\mathbf{R}})$ associated with $X_u' = \mathcal{U}X_u$.

**Problem 3.** Consider the flow associated to the planar ODE system ($\alpha \in \mathbf{R}$ parameter)

$$\begin{cases} \dot{x} = \alpha x + y + xy + 4x^2 + 8xy^2 \\ \dot{y} = -x + \alpha y - x^2 + 2y^2 \end{cases} \tag{3}$$

a) Show that the origin is an equilibrium point of system (3) for any $\alpha \in \mathbf{R}$. Then prove that there exists a value $\alpha^*$ such that $(0,0)$ is a nonhyperbolic equilibrium of (3) with $\alpha = \alpha^*$ and characterize the stability and type of $(0,0)$ for $\alpha \neq \alpha^*$.

b) Use the complex coordinate $z = x + iy$ to transform system (3) in an equation of the form

$$\dot{z} = (\mu(\alpha) + i\omega(\alpha))z + \frac{1}{2}g_{20}z^2 + g_{11}z\bar{z} + \frac{1}{2}g_{02}\bar{z}^2 + \frac{1}{6}g_{30}z^3 + \frac{1}{2}g_{21}z^2\bar{z} + \frac{1}{2}g_{12}z\bar{z}^2 + \frac{1}{6}g_{03}\bar{z}^3.$$

Hint: The fastest way to calculate the equation in $z$ is by writing $\dot{z} = \dot{x} + i\dot{y}$ then use $x = (z + \bar{z})/2$ and $y = (z - \bar{z})/(2i)$ directly in system (3).

c) Determine the (complex) coefficients $g_{jk}$ (that may depend on $\alpha$) and apply the formula below to compute the **first Lyapunov coefficient** $l_1(\alpha^*)$:

$$l_1(\alpha^*) = \frac{1}{2\omega_0^2} Re(\, i\, g_{20}^*\, g_{11}^* + \omega_0\, g_{21}^*\,)$$

where $\omega_0 = \omega(\alpha^*)$ and $g_{jk}^* = g_{jk}(\alpha^*)$, then prove that system (3) undergoes a Hopf bifurcation at $\alpha = \alpha^*$.

d) Use the results in (b) and (c) to determine if the bifurcation is supercritical or subcritical, then answer the following two questions (justify your answers):

– Is the limit cycle born through the Hopf bifurcation, stable or unstable?

– On which side of the bifurcation point ($\alpha > \alpha^*$ or $\alpha < \alpha^*$) does the limit cycle appear?

**Problem 4.**

a) Show that the following functions define a flow on the space indicated:

$$\varphi_t(x) = \frac{x + \tanh(t)}{1 + x\tanh(t)}, \qquad x \in [-1, 1]. \tag{4}$$

b) Find the vector field associated with the flow defined by (4).

c) Show that the flows associated with ODE $X' = AX$ and $Y' = BY$, respectively, where $X, Y \in \mathbf{R}^2$ and

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}, \qquad B = \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix}$$

are $\varphi_t(x_1, x_2) = \left(e^{-2t}x_1, e^{-2t}x_2\right)$ and $\psi_t(y_1, y_2) = \left(e^{-2t}(y_1 + ty_2), e^{-2t}y_2\right)$.

d) Find $Y = h(X)$, or $(y_1, y_2) = h(x_1, x_2) = (\, h_1(x_1, x_2), h_2(x_1, , x_2)\,)$ a **topological conjugacy** between the flows $\varphi$ and $\psi$ from (c).

Hint: Look for $h = (h_1, h_2)$ in the form of: $h_1(x_1, x_2) = x_1 + f(x_2)$ and $h_2(x_1, x_2) = h_2(x_2)$. Make sure you verify all properties that $h$ needs to satisfy in order to be a topological conjugacy!