Name:____________________________________

Ph.D. Qualifying Exam in PDE

January 2024

This booklet contains five problems with equal weights on seven pages of paper including this cover and the extra blank page on the back.

Please solve any four problems from five problems in this PDE part of the exam.

You can use the back of the paper and the extra paper on the back, if extra space is needed.

Note to Proctor: Please print the exam one-sided on seven pages of paper.

1. Suppose u is a harmonic function defined on $\{(x,y) : 0 < x^2 + y^2 < 1\}$ and suppose that $|u| \le 1$. Prove that u can be extended continuously to the origin so that the extension is harmonic in the whole disk.

2. Let B_1 be the unit ball in $\mathbf{R}^n$.

   (i) Define $H^1(B_1)$ and define $H_0^1(B_1)$.

   (ii) For $f \in L^2(B_1)$, define a weak solution to

$$\begin{cases} -\Delta u &= f, \quad \text{in } B_1, \\ u &= 0, \quad \text{on } \partial B_1. \end{cases}$$

3. (i) Show that there is a unique weak solution for the equation in Problem 2(ii).

   (ii) Prove there is a constant C so that $\|u\|_{L^2(B_1)} \le C\|f\|_{L^2(B_1)}$.

4. Consider the heat equation on the whole x-axis. Show that there is a unique solution $u = u(x,t)$ that is linear growth in x to

$$\begin{cases} u_t - u_{xx} &= 0, \quad x \in \mathbb{R}, \quad 0 < t < \infty, \\ u(x,0) &= x^+, \text{(the plus part)}. \end{cases}$$

5. Find a weak solution each to the nonlinear conservation law with the following Riemann type initial data such that the solution satisfies the entropy condition:

   $u_t + (\frac{u^2}{2})_x = 0, \quad x \in \mathbb{R}, \ t > 0,$

   (i) with initial data

$$u(x,0) = \begin{cases} 2 & x < 0, \\ 1 & x \ge 0; \end{cases}$$

   and

   (ii) with initial data

$$u(x,0) = \begin{cases} 1 & x < 0, \\ 2 & x \ge 0. \end{cases}$$