Ph.D. and M.S. Qualifying Exam in Algebra-Winter 2025

January 16, 9:00am-12:00pm, in room 118 MLH

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Rules of the exam

- You have 180 minutes to complete this exam.
- The exam contains 4 sections (group theory, ring theory, linear algebra and module theory, and field theory), each consisting of 4 problems. Please submit solutions for exactly 2 problems, at your choice, from each section.
- Please mark the problems to be graded on the first column of the grading table on page 2.
- Show all your work; any answer without an explanation will get you zero points.
- Please read the questions carefully; some ask for more than one thing.
- Do not forget to write your name, see page 2.
- During the exam you are not allowed to use a calculator, cell phone, ipad, laptop, computer or any other electronic or internet browsing device.

Good luck!

NAME (*PRINT*): ________________________________

Mark in the first column below which problems should be graded!

Your Choice Problem	**Points**	**Your Score**
	20	
	20	
	20	
	20	
	20	
	20	
	20	
	20	
Total	160	

Group theory

G - I: Let G be a finite group, p be a prime number and S be a Sylow p-subgroup of G. Let $X, Y \subseteq Z(S)$ for which there exists $g \in G$ such that $Y = gXg^{-1}$. Show that there exists $n \in N_G(S)$ such that $nxn^{-1} = gxg^{-1}$ for all $x \in X$.
Make sure you include all the details in your answer!

G - II: An action of a group G an a set X is called *2-transitive* if for every $(x_1, y_1), (x_2, y_2) \in X \times X$ with $x_1 \neq y_1$ and $x_2 \neq y_2$ one can find $g \in G$ such that $gx_1 = x_2$ and $gy_1 = y_2$. Show that if G acts 2-transitively on a set X with at least two elements then for every $x \in X$ its stabilizer $G_x < G$ is a maximal proper subgroup.
Make sure you include all the details in your proof.

G - III: Solve BOTH of the following problems:

1. Show that subgroups of finitely generated abelian groups are finitely generated. Please include a detailed proof, you are not allowed just to quote a theorem in algebra.
2. If $G \leqslant \mathbb{Z}^3$ is the subgroup generated by $(1,2,5)$, $(3,7,1)$ and $(2,1,2)$ then determine the isomorphism class of $\mathbb{Z}^3/G$.

   Make sure you include all the details in your answer!

G - IV: Solve ONE of the following:

1. How many subgroups of index 2 does a free group with 2025 generators have?

   Make sure you include all details in your proof!
2. Show that every group generated by two elements of order two is solvable.

   Make sure you include all details in your proof!

Ring theory

R - I: Let R be a ring with identity and let n be a positive integer. Prove that every two-sided ideal of $\mathbb{M}_n(R)$ is equal to $\mathbb{M}_n(J)$ for some two-sided ideal J of R. Make sure you include all the details in your proof.

R - II:

1. State Eisenstein's irreducibility criterion for primitive polynomials over a UFD.
2. Decide whether the following two polynomials are irreducible or not over $\mathbb{Z}[x]$.
   - $P(x) = x^{277} + 277x^{276} + 277x^{275} + 555$
   - $Q(x) = (x^2 + 1)^n + 7$ for some positive integer n.

R - III: Let R be a commutative ring. Let $\emptyset \neq S \subseteq R$ be a saturated multiplicative set, i.e. if $a, b \in R$, then $ab \in S$ if and only if $a, b \in S$. Show that $R \setminus S$ is a union of prime ideals. Make sure you prove all the details in your answer!

R - IV: (True-False) If F be a field such that its multiplicative group $F^{\times} = F \setminus \{0\}$ is finitely generated then F is finite.
If you believe it is true, provide a complete proof. If you believe it is false, provide a concrete counterexample or prove in some fashion that the statement does not hold.

Modules and linear algebra theory

MLA - I: Let F be a field and consider the direct sum of cyclic $F[x]$-modules

$$V = \frac{F[x]}{(x+1)^2} \oplus \frac{F[x]}{(x^2-1)} \oplus \frac{F[x]}{(x-1)^2}.$$

1. Determine the invariant factors and the elementary divisors of V as an $F[x]$-module.
2. Determine the rational canonical form of the linear transformation $T : V \to V$ given by multiplication by x.

Make sure you include all the details including stating clearly all results you use in your proof!

MLA - II: Let $T, S_1, S_2 \in \mathbb{M}_n(\mathbb{C})$ such that S_1, S_2 are normal and $TN_1 = N_2T$. Show that $TN_1^* = N_2^*T$.
Recall that given any matrix $(a_{ij})_{i,j} = A \in \mathbb{M}_n(\mathbb{C})$, its *conjugate transpose* is defined as $A^* = (\overline{a_{ji}})_{1 \leq i,j \leq n}$. Also A is called *normal* if $AA^* = A^*A$.

MLA - III: An abelian (additive) group Q is called *divisible* if and only if for every positive integer n and every $y \in Q$ there is $x \in Q$ such that $nx = y$. Assume that $A \leqslant B$ are abelian groups and Q is a abelian divisible group. Show that every group homomorphisms $\psi : A \to Q$ extends to a group homomprohism $\tilde{\psi} : B \to Q$.
For the proof you are not allowed to just quote a theorem but instead you have to show all details!

MLA - IV: Let R be a ring with identity and let M be a projective left R-module. Show that one can find elements $\{m_i \,|\, i \in I\} \subset M$ and R-modules homomorphisms $f_i : M \to R$ for $i \in I$ such that for every $m \in M$ the following two relations hold:

1. $f_i(m) \neq 0$ for any finitey many $i \in I$;
2. $m = \sum_{i \in I} f_i(m) m_i$.

For the proof you are not allowed to just quote a theorem but instead you have to show all details!

Fields and Galois theory

F - I: Solve at your choice ONE of the following problems:

1. Show that $\mathbb{Q}(\sqrt{2+\sqrt{2}})/\mathbb{Q}$ is a Galois extension of degree 4 with cyclic Galois group.

   Make sure you include all the details including stating clearly all results you use in your proof!

2. Compute the Galois group of the polynomial $x^5 - 9x + 3$ over $\mathbb{Q}$.

   Make sure you include all the details including stating clearly all results you use in your proof!

F - II: Let $K = \mathbb{Q}(\sqrt[n]{a})$, where $a \in \mathbb{Q}_+$ and suppose that $[K : \mathbb{Q}] = n$. Let E be a subfield of K and let $[E : \mathbb{Q}] = d$. Prove that $E = \mathbb{Q}(\sqrt[d]{a})$.
Make sure you include all details and state clearly all results you use in your proof!

F - III: Let K/F be an extension of finite fields. Define the norm $N_{K/F} : K \to F$ and show it is surjective.
Make sure you include all details in your proof!

F - IV: Let F be a field, and let $f, g \in F[x] \setminus \{0\}$ be relatively prime and not both constant. Show that $F(x)$ has finite degree, $d = \max(deg(f), deg(g))$, over its subfield $F(\frac{f}{g})$.
Make sure you include all details in your proof!

Answers

Your Choice

G-

Solution:

Your Choice

G-

Solution:

Your Choice

R-

Solution:

Your Choice

R-

Solution:

Your Choice

MLA-

Solution:

Your Choice

MLA-

Solution:

Your Choice

F-

Solution:

Your Choice

F-

Solution:

Scratch Paper

Scratch Paper

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