Math@Iowa Analysis Qualifying Exam
Friday, January 17, 2025
9:00 am - 12:00 noon in Room 218 MLH

Examination Committee: Professors Raúl E. Curto and Palle E.T. Jørgensen

Your Name:

Instructions

1. This is a closed-book exam; no books or notes are allowed. This exam includes a Prep Sheet (attached), which you may use as a reference.
2. You are not allowed to use any internet resources; use of calculators is not allowed.
3. This Qualifying Exam consists of 10 questions, broken down into two parts of 5 questions each, for a maximum score of 70 points. Please submit **a total of seven question, with at least three chosen from each Part**. Please summarize your choices in the table below, by placing a ✓ inside the box located next to each choice.

5. Please write your answers within the space provided for each question. If at all possible, please write your answers in ink (blue, black or red). However, if writing math using a pencil is what you always do, by all means use a pencil for this exam too.
6. **For each solution you provide, make sure to include all the details, and state correctly all the known results you use. If appropriate, feel free to refer to the attached Prep Sheet.**
7. A score in the range **63 – 70** guarantees a **Ph.D. Pass**; a score in the range **56 – 62** guarantees an **M.S. Conditional Pass**.
8. **Before you start, please write your name on the space provided in this page. When you finish, please turn in this exam booklet.**

☐ R-1A	☐ R-1B			☐ C-1A	☐ C-1B
☐ R-2				☐ C-2A	☐ C-2B
☐ R-3A	☐ R-3B			☐ C-3	
☐ R-4				☐ C-4A	☐ C-4B
☐ R-5A	☐ R-5B			☐ C-5A	☐ C-5B

PART I: Real Analysis

(10 points) **(R-1)**

Choose either Problem R-1A or Problem R-1B below and solve it.

Problem R-1A. Let $E := (0, +\infty)$, $g \in L^1(E)$, and define $f : E \to \mathbb{R}$ as

$$f(x) := \int_E g(t)dt.$$

(a) Prove that f is absolutely continuous on E.
(b) Prove that f may fail to be Lipschitz on E; that is, find $g \in L^1(E)$ such that the associated f is not Lipschitz on E.
(c) Prove that, when $g \in L^\infty(E)$, then f is Lipschitz.
(d) For g as in (c), find the best Lipschitz constant for f.

Problem R-1B. Determine if the following statement is true or false. If true, provide a proof; if false, provide a counterexample or show in some fashion why the statement is false. In either case, feel free to use results from the Prep Sheet.

(True or False) Let $\mathcal{CS}$ be the collection of all closed subsets of $\mathbb{R}$. Then $\mathcal{CS}$ is equipotent to $\mathbb{R}$; that is, there exists a bijection between $\mathcal{CS}$ and $\mathbb{R}$.

Problem R-1 cont.

(10 points) **(R-2)** Let $[a,b]$ be a closed, bounded interval and $C[a,b]$ be normed by the maximum norm. Let T be a bounded linear functional on $C[a,b]$. For $x \in [a,b]$, let g_x be the member of $C[a,b]$ that is linear on $[a,x]$ and on $[x,b]$ with $g_x(a) = 0, g_x(x) = x - a$ and $g_x(b) = x - a$. Define $\Phi : [a,b] \longrightarrow \mathbb{R}$ by

$$\Phi(x) := T\left(g_x\right) \quad (x \in [a,b]).$$

Show that Φ is Lipschitz on $[a,b]$.

Problem R-2 cont.

(10 points) **(R-3)**

Choose either Problem R-3A or Problem R-3B below and solve it.

Problem R-3A. Consider the Banach spaces $L^1[0,1]$ and $L^\infty[0,1]$. Prove that there does **not** exist a **continuous and surjective** linear map $T : L^1[0,1] \longrightarrow L^\infty[0,1]$.

Problem R-3B. Recall that for A and B nonempty subsets of $\mathbb{R}$, we define

$$d(A, B) := \inf\{d(x, B) : x \in A\},$$

where

$$d(x, B) := \inf\{|x - y| : y \in B\}.$$

Recall, too, that the outer measure m^* is countably subadditive. For arbitrary sets $A, B \subseteq \mathbb{R}$, prove that

$$d(A, B) > 0 \Longrightarrow m^*(A \cup B) = m^*(A) + m^*(B).$$

Problem R-3 cont.

(10 points) **(R-4)** On a closed, bounded nondegenerate interval $[a, b]$, consider a sequence $\{f_n\}$ of increasing, absolutely continuous functions on $[a, b]$. Assume:

(a) $f_n(a) = 0$ for all $n \in \mathbb{N}$;

(b) $f_n' \leq f_{n+1}'$ a.e. on $[a, b]$, for all $n \in \mathbb{N}$;

(c) the sequence $\{f_n(b)\}$ is bounded.

Prove:

(i) there exists $f : [a, b] \to \mathbb{R}$ such that $\{f_n\} \to f$ pointwise on $[a, b]$;

(ii) f is absolutely continuous on $[a, b]$; and

(iii) $\{f_n\}$ converges to f uniformly on $[a, b]$.

Problem R-4 cont.

(10 points) **(R-5)**

Choose either Problem R-5A or Problem R-5B below and solve it.

Problem R-5A. Determine if the given statement is true or false. If true, provide a proof; if false, provide a counterexample or show in some fashion why the statement is false. In either case, feel free to use results from the Prep Sheet.

For the closed interval $[0,1]$, let $\mathcal{P}(1;0)$ denote the collection of polynomials in the indeterminate t which vanish at the point $t=1$.

(True or False) $\mathcal{P}(1;0)$ is dense in $L^1[0,1]$; that is, every $f \in L^1[0,1]$ can be approximated in the norm $\|\cdot\|_1$ by polynomials in $\mathcal{P}(1;0)$.

Problem R-5B. (Embedding of ℓ^p in $L^p[1,\infty)$). Let $1 \le p < \infty$. For $a = (a_1, a_2, \ldots) \in \ell^p$, define $f_a : [1,\infty) \to \mathbb{R}$ by

$$f_a(x) := \sum_{k=1}^{\infty} a_k \chi_{[k,k+1)}(x) \quad (x \in [1,\infty)).$$

(B_1) Prove that $f_a \in L^p[1,\infty)$.

(B_2) Prove that $\|f_a\|_p = \|a\|_p$.

(B_3) Prove that the map $T : \ell^p \to L^p[1,\infty)$ given by $T(a) := f_a$ is linear and **injective**, but **not** surjective.

(B_4) Prove that the range of T is **not** dense in $L^p[1,\infty)$.

Problem R-5 cont.

PART II: Complex Analysis

(10 points) **(C-1)**

Choose either Problem C-1A or Problem C-1 below and solve it.

Problem C-1A. Let $\mathbb{D}$, $\overline{\mathbb{D}}$, and $\mathbb{T}$ denote the open unit disc, the closure of the open unit disc, and the unit circle, respectively. Let $f : \mathbb{T} \to \mathbb{C}$ be a continuous function, and define $\hat{f} : \overline{\mathbb{D}} \to \mathbb{C}$ by

$$\hat{f}(z) := f(z) \quad (z \in \mathbb{T})$$

and

$$\hat{f}(re^{i\theta}) := \frac{1}{2\pi} \int_{-\pi}^{\pi} f\left(e^{it}\right) P_r(\theta - t) dt \quad (r < 1),$$

where P_r denotes de Poisson kernel.

Given $r < 1$, define $\hat{f}_r : \mathbb{T} \to \mathbb{C}$ by

$$\hat{f}_r(z) := \hat{f}(rz) \quad (z \in \mathbb{T}).$$

Show that for each $r < 1$ there is a sequence $\{p_n(z, \bar{z})\}$ of polynomials in z and $\bar{z}$ such that $p_n(z, \bar{z}) \to \tilde{f}_r(z)$ uniformly for z in $\mathbb{T}$. (Hint: Use Definition 2.1 in Prep Sheet.)

Problem C-1B. Determine if each of the given statements is true or false. If true, provide a proof; if false, provide a counterexample or show in some fashion why the statement is false. In either case, feel free to use results from the Prep Sheet.

Let $\mathbb{D}$ be the open unit disk in $\mathbb{C}$.

(i) (True or False) There exists an analytic function $f : \mathbb{D} \to \mathbb{D}$ such that $f(\frac{1}{2}) = \frac{2}{3}$ and $f'(\frac{1}{2}) = \frac{7}{9}$.

(ii) Let $f : \mathbb{D} \to \mathbb{D}$ be an analytic function. Assume that f has two distinct fixed points in $\mathbb{D}$.

(True or False) f is the identity function; that is, $f(z) = z$ for every $z \in \mathbb{D}$.

Problem C-1 cont.

(10 points) **(C-2)**

Choose either Problem C-2A or Problem C-2B below and solve it.

Problem C-2A. Determine if the given statement is true or false. If true, provide a proof; if false, provide a counterexample or show in some fashion why the statement is false. In either case, feel free to use results from the Prep Sheet.

Let $\mathbb{D}$ be the open unit disc in $\mathbb{C}$. For $0 < a < b < 1$, let $G(a, b)$ denote the elliptical region $\{w \equiv (u, v) \in \mathbb{C} : \frac{u^2}{a^2} + \frac{v^2}{b^2} < 1\}$.

(True or False) The transformation $(\text{Re } z, \text{Im } z) \mapsto (a \cdot \text{Re } z, b \cdot \text{Im } z)$ establishes a conformal equivalence f between $\mathbb{D} \setminus \{(0, 0)\}$ and $G(a, b) \setminus \{(0, 0)\}$; that is, f is analytic, injective, and surjective.

Problem C-2B. (a) Show that the function

$$z \mapsto z + \frac{1}{z}$$

is an analytic isomorphism of the region outside the unit circle onto the plane from which the segment $[-2, 2]$ has been deleted.

(b) What is the image of the unit circle under this mapping? (Hint: Use polar coordinates.)

(c) Show that the circle $r = c$ with $c > 1$ maps to an ellipse with major axis $c + \frac{1}{c}$ and minor axis $c - \frac{1}{c}$.

Problem C-2 cont.

(10 points) **(C-3)** **Use complex analysis methods** to show that

$$\int_0^\infty \frac{\cos(2x)}{(1+x^2)^2}\,dx = \frac{3\pi}{4e^2}.$$

Problem C-3 cont.

(10 points) **(C-4)**

Choose either Problem C-4A or Problem C-4B below and solve it.

Problem C-4A. Consider the square-root mapping $w = \sqrt{z}$, defined in the region

$$G := \mathbb{C} \setminus \{z \in \mathbb{R} : z \leq 0\}.$$

Describe in complete detail the image under this mapping of the horizontal strip

$$S := \{z \in \mathbb{C} : 0 < \operatorname{Im} z < 1\}.$$

Problem C-4B. Let $\overline{\mathbb{D}}$ denote the closure of the open unit disc in $\mathbb{C}$. Let G be an open set in $\mathbb{C}$, and assume that $\overline{\mathbb{D}} \subseteq G$. Let f be analytic in G and assume that $|f(z)| < 1$ for $|z| = 1$. Show that there is a unique z with $|z| < 1$ and $f(z) = z$.

(Hint: Let $g(z) := f(z) - z$ and $h(z) := z$. Now use Rouché's Theorem.)

Problem C-4 cont.

(10 points) **(C-5)**

Choose either Problem C-5A or Problem C-5B below and solve it.

Problem C-5A. Let $\mathbb{D}$ be the open unit disc in $\mathbb{C}$. Find a concrete description of a Möbius transformation $T(z) = \dfrac{az+b}{cz+d}$ such that $T(\mathbb{D}) = \mathbb{D}$ and $T(\frac{i}{2}) = 0$.

Problem C-5B. On the space $H(\mathbb{D})$ of analytic functions on the open unit disc, consider the Hilbert and Cesàro transforms, defined (respectively) as

$$H(f)(z) := \int_0^1 \frac{f(t)}{1-tz}\,dt \quad (f \in H(\mathbb{D}), z \in \mathbb{D})$$

and

$$C(f)(z) := \frac{1}{z} \cdot \int_0^z \frac{f(t)}{1-t}\,dt \quad (f \in H(\mathbb{D}), 0 \neq z \in \mathbb{D}).$$

(a) Given the sequence of functions $e_n(z) := z^n$ $(z \in \mathbb{D})$, where n is a positive integer, prove that

$$C(e_n) = e_n H(e_n) \quad (\text{for all } n \geq 1).$$

(b) Use the above identity to show that for every $n \geq 1$, the function $C(e_n)$ has a removable singularity at $z = 0$.

Problem C-5 cont.

Additional Space

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End of Exam