Ph.D. Qualifying Exam in Topology

Ben Cooper, Mohammad Tehrani

January, 2025

Instructions.

- Do eight problems: four from part A and four from part B.
- This is a closed book examination: you should have no books, technology or paper of your own. Paper will be provided by the test center.
- Please do your work on the paper provided according to the format outlined below.
  - On each page of your solutions
    - Write your name
    - Write the page number
    - Indicate which problem is being addressed
  - When you start a new problem, start a new page
  - Only write on one side of the paper
  - Make a cover page and indicate which eight problems you want graded.
- Always justify your answers unless explicitly instructed otherwise.
- You may use theorems if the problem is not a step in proving that theorem. You must to state any theorems that you use clearly and carefully.

Part A - Algebraic Topology

In the problems below, the symbols for the disk D^n, the sphere S^n and the simplex Δ^n can be understood to mean the subspaces below

$$D^n := \{v \in \mathbb{R}^n : |v| \leq 1\} \qquad S^n := \{v \in \mathbb{R}^{n+1} : |v| = 1\}$$
$$\Delta^n := \{(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} : \sum_{i=0}^{n} x_i = 1\}$$

1. Let $T := S^1 \times S^1$ be the torus, $\gamma(t) := (e^{2\pi it}, 1)$ the loop in T. Pick a point $p \in T\backslash\gamma(S^1)$ and let $\eta : [0,1] \to T^2$ parameterize the boundary ∂D_η of a disk $D_\eta := \{v \in T^2 : |v - p| \leq \epsilon\}$ which is sufficiently small[1]. Let S be the quotient space formed by identifying the two circles η and γ in $T\backslash int(D_\eta)$:

   $$S := (T\backslash int(D_\eta)) \,/ \sim \qquad \text{where} \qquad \gamma(t) \sim \eta(t) \text{ for } 0 \leq t \leq 1.$$

   Compute $\pi_1(S, \eta(0))$.

2. Let $\Delta_{n,k} := \Delta/\sim$ be the standard n-simplex Δ with all of it's k-dimensional subsimplices identified:

   $$\Delta_{n,k} := \Delta/\sim \qquad \text{where} \qquad [i_0, i_1, \ldots, i_k] \sim [j_0, j_1, \ldots, j_k]$$

   where $[i_0, i_1, \ldots, i_k] := \{(x_0, x_1, \ldots, x_n) \in \Delta : x_i = 0 \text{ if } i \notin \{i_0, i_1, \ldots, i_k\}\} \subseteq \Delta$ and $\sim$ is applied to all distinct pairs of sets $\{i_0, i_1, \ldots, i_k\} \in \{0, 1, \ldots, n\}^{\times(k+1)}$ such that $i_0 < i_1 < \cdots < i_n$.

   Compute $H_1(\Delta_{n,k})$ for all $0 \leq k < n$.

3. (a) Let the space $X := T\backslash int(D_\eta)$ and the curve γ be as in Problem 1 above. Construct the covering space $p_\gamma : X_\gamma \to X$ associated to the subgroup generated by γ.

   (b) Identify the deck transformation group $Deck(X_\gamma/X)$ and construct it's action $a : Deck(X_\gamma/X) \times X_\gamma \to X_\gamma$.

[1] The value of ϵ is small enough for there to exist an embedding $\varphi : D^2 \to T\backslash\gamma(S^1)$ such that $im(\varphi) = D_\eta$ so that $\eta(t) = \varphi(e^{2\pi it})$.

4. Let (C, d) be a chain complex of F-vector spaces for some field F:

$$\cdots \to C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} C_{n-2} \to \cdots$$

such that $d_{n-1}d_n = 0$ for $n \in \mathbb{Z}$. If we assume $C_n = 0$ for all but finitely many $n \in \mathbb{Z}$ then the Euler characteristic $\chi(C) \in \mathbb{Z}$ is the integer given by

$$\chi(C) := \sum_{i \in \mathbb{Z}} (-1)^i \dim_F C_i.$$

Prove that the Euler characteristic is equal to the alternating sum of dimensions of homology groups:

$$\chi(C) = \sum_{i \in \mathbb{Z}} (-1)^i \dim_F H_i(C, d)$$

5. Let $x, y \in D^2$ be two distinct points in the 2-dimensional disk and D_x, D_y be small disjoint closed balls about x and y respectively. Set

$$D_2 := D^2 \backslash int(D_x \sqcup D_y)$$

Let $pt \in \partial D^2 \subset D_2$ be a basepoint and let $f : (D_2, pt) \to (D_2, pt)$ be a homeomorphism which fixes the basepoint.

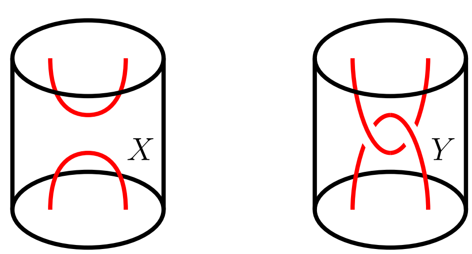
   (a) Compute the fundamental group $\pi_1(D_2, pt)$

   (b) Compute the fundamental group of the mapping torus $\pi_1(M_f, (pt, 0))$ where

$$M_f = D_2 \times [0, 1] / \left((x, 0) \sim (f(x), 1)\right)$$

6. Consider the spaces $A := D^2 \times [0, 1] \backslash X$ and $B := D^2 \times [0, 1] \backslash Y$ pictured below. Let $C := \partial(D^2 \times [0, 1])$. Prove that there is no homeomorphism

$$f : A \to B \qquad \text{such that} \qquad f|_C = 1_C$$



Part B - Manifolds and vector bundles

1. Give an example of a one-to-one immersion $f\colon M \longrightarrow N$ which is not an embedding. If M is compact, show that a one-to-one immersion $f\colon M \longrightarrow N$ is an embedding.

2. Prove that $\mathrm{O}(n) = \{A \in M_{n\times n}(\mathbb{R}) : A^T A = \mathrm{Id}\}$ is a manifold and find its dimension.

3. Suppose $\gamma : S^1 \to \mathbb{R}^2$ is a smooth embedding. Compute the integral $\int_\gamma \theta$ when
$$\theta = xy^2dx + x^2ydy$$

4. If $f\colon M \longrightarrow N$ is a smooth map between equi-dimensional closed oriented manifolds, degree f of is the integer
$$\deg(f) = \frac{\int_M f^*\omega}{\int_N \omega}$$
where ω is any top-form N with $\int_N \omega \neq 0$ (it can be shown that the right-hand side is independent of the choice of ω and is an integer). For $m, n \geq 1$, show that any smooth map
$$f\colon S^{m+n} \longrightarrow S^m \times S^n$$
has degree 0. (Bonus: Is there a non-zero degree map $S^m \times S^n \longrightarrow S^{m+n}$?)

5. Show that the restriction ω of the 2-form
$$xdy \wedge dz - ydx \wedge dz + zdx \wedge dy$$
to S^2 is a volume form. Compute the integral
$$\int_{S^2} \omega$$
Show that the vector field
$$x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$$
on $\mathbb{R}^3$ is tangent to S^2 and thus defines a vector field X on S^2. Calculate the divergence of X with respect to ω

6. Prove that every oriented real vector bundle of rank 2 on a manifold can be turned into a complex vector bundle (of rank 1) such that multiplication by $\mathfrak{i}$ is an orientation-preserving automorphism of the vector bundle.