Ph.D. qual. exam./M.S. comp. exam. on Numerical analysis.
January 2025.

There are two Parts. **In Part I, answer at least 5 out of 8 questions. In Part II, answer at least 5 out of 8 questions.** Each question is worth 10 points. Only the best 5 of each Part will count. Always **show your calculations and justify your answers**.

Part I, answer at least 5 out of 8 questions:

1. (a) Give the Taylor polynomial $P_3(x)$ of degree 3 about $a = 0$ of the function $f(x) = \ln(1 - 2x)$.
   (b) What is the relative condition number of the function $f(x) = \ln(1 - 2x)$ at $x = 3 \cdot 10^{-8} = 0.00000003$?
   (c) Give at least 10 correct significant decimal digits of
   $$g(x) = \frac{\ln(1 - 2x) + 2x}{x^2}$$
   at $x = 3 \cdot 10^{-8} = 0.00000003$. Hint: do not evaluate this expression directly on your calculator.
   (d) Explain succinctly why the calculator gives an inaccurate numerical result when evaluating $g(x)$ directly at $x = 3 \cdot 10^{-8}$.

2. Consider the interpolation polynomial $P_n(x)$ of the function $f(x) = \cos(x)$ on $[0, \pi/2]$ at the $n+1$ Chebyshev nodes
   $$x_j = \frac{\pi}{4} + \frac{\pi}{4} \cos\left(\frac{(2(n-j)+1)}{2n+2}\pi\right) \quad \text{for} \quad j = 0, \ldots, n.$$
   Find the smallest value of n ensuring that the absolute error on $[0, \pi/2]$ of the interpolation polynomial is less than 10^{-6}, i.e., find the smallest n such that
   $$\|f - P_n\|_\infty = \max_{x \in [0,\pi/2]} |\cos(x) - P_n(x)| \leq 10^{-6}.$$
   Remark: you are NOT asked to give the interpolation polynomial $P_n(x)$.

3. Is the following function on the interval $[0, 2]$ a cubic spline? If yes is it a periodic spline?
   $$s(x) = \begin{cases} 8 + 2x & \text{for} \quad x \in [0, \frac{1}{2}], \\ 7 + 8x - 12x^2 + 8x^3 & \text{for} \quad x \in [\frac{1}{2}, 2]. \end{cases}$$

4. For $s = 3$ find the weights b_1, b_2, b_3 and nodes c_1, c_2, c_3 of a symmetric(!) quadrature formula
   $$h_j\left(b_1 f(x_j + c_1 h_j) + b_2 f(x_j + c_2 h_j) + b_3 f(x_j + c_3 h_j)\right) \approx \int_{x_j}^{x_j + h_j} f(x)dx$$
   of highest possible order satisfying
   $$c_1 = \frac{1}{4}.$$
   What is its order?

5. Consider approximating the integral
   $$I := \int_0^{\sqrt{2}} e^{-x^2} dx$$
   with the left-rectangle rule using an equidistant subdivision of $[0, \sqrt{2}]$ in n subintervals. Find a value of n that ensures that the absolute total error of this approximation is less than 10^{-4}, i.e., such that
   $$|I - R_n| = \left| \int_0^{\sqrt{2}} e^{-x^2} dx - h \sum_{j=0}^{n-1} e^{-(jh)^2} \right| \leq 10^{-4}.$$

6. Compute the Fourier coefficients a_k, b_k, $k = 0, 1, 2, \ldots$ for the periodic function $f(t) = |t|$ on the interval $[-\pi, \pi]$ repeated periodically with period 2π. What is the trigonometric polynomial $S_n(t)$ for $n = 7$ minimizing $\int_0^{2\pi} (f(t) - S_n(t))^2 dt$?

7. Find the polynomial $q_2(x)$ of degree $\deg(q_2) \leq 2$ approximating the function $f(x) = x^{4/3}$ on the interval $[-1, 1]$ which minimizes

$$\int_{-1}^{1} (f(x) - q_2(x))^2 dx.$$

8. Define Newton's method and the secant method to find a zero to a scalar nonlinear equation $g(x) = 0$ with $g : [a, b] \to \mathbb{R}$. Give an advantage of the secant method compared to Newton's method.

Part II, answer at least 5 out of 8 questions:

1. In term of matrices, what is the Cholesky decomposition of a real symmetric positive definite matrix $A \in \mathbb{R}^{n \times n}$ (do not give the details on how to find it)? Give the Cholesky decomposition of the matrix

$$A = \begin{bmatrix} 4 & -4 & 2 \\ -4 & 5 & -3 \\ 2 & -3 & 11 \end{bmatrix}.$$

2. To solve approximately a system of linear equations $Ax = b$ with $A \in \mathbb{R}^{n \times n}$ and $b \in \mathbb{R}^n$ explain what Gauss-Seidel iterations are. Consider the system of linear equations

$$\begin{bmatrix} 4 & 1 \\ 8 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}. \qquad (1)$$

Starting from the initial vector

$$x^{(0)} = \begin{bmatrix} x_1^{(0)} \\ x_2^{(0)} \end{bmatrix} := \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

do Gauss-Seidel iterations converge to the solution x^* of the system of linear equations (**??**), i.e., do we have $\lim_{k\to\infty} x^{(k)} = x^*$? Prove convergence or divergence (computing the first few iterates is not a proof).

3. Consider the linear least squares problem

$$\min_{x \in \mathbb{R}^n} \|b - Ax\|_2 \qquad (2)$$

where $A \in \mathbb{R}^{m \times n}$ with $m \geq n$, $b \in \mathbb{R}^m$, and $x \in \mathbb{R}^n$.

(a) Give the normal equations that a minimizer $x^\star$ of (**??**) must satisfy.

(b) Suppose $A \in \mathbb{R}^{m \times n}$ is of rank n and that we have a QR decomposition of A with $Q \in \mathbb{R}^{m \times m}$ orthogonal and $R \in \mathbb{R}^{m \times n}$ upper triangular, show how to obtain a minimizer $x^\star$ of (**??**) using this QR decomposition of A.

(c) Find $x \in \mathbb{R}^2$ minimizing $\|b - Ax\|_2$ for

$$A := \begin{bmatrix} 1 & -2 \\ 2 & 1 \\ 1 & 1 \end{bmatrix} \in \mathbb{R}^{3 \times 2}, \qquad b := \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix} \in \mathbb{R}^3.$$

4. Consider the following nonlinear system of ODEs

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \cos(t x_1) \\ e^{-3t} x_1^2 \end{bmatrix}.$$

Give the value $x_1 = (x_{11}, x_{12})$ after one step with stepsize $h = 0.2$ of the Taylor series method of order 2 starting from $x_0 = (1, 2)$ at $t_0 = 0$.

5. (a) Consider a system of ODEs $\dot{x} = f(t, x)$ and the following explicit Runge-Kutta method

$$\begin{aligned} X_1 &= x_0 \\ X_2 &= x_0 + h\frac{1}{3}f(t_0, X_1) \\ X_3 &= x_0 + h(-f(t_0, X_1) + 2f(t_0 + h/3, X_2)) \\ x_1 &= x_0 + h\left(\frac{3}{4}f(t_0 + h/3, X_2) + \frac{1}{4}f(t_0 + h, X_3)\right) \end{aligned}$$

Give the coefficients $(b_1, b_2, b_3; c_1, c_2, c_3; a_{21}, a_{31}, a_{32})$ of this explicit Runge-Kutta method (preferably in a Butcher tableau).

(b) Find the local order p of this method.

(c) What is its stability function $R(z)$?

6. We consider the following explicit linear multistep method with stepsize h applied to $\dot{x} = f(t, x)$ (using the notation $f_j := f(t_j, x_j)$)

$$x_{n+1} = 3x_n - 2x_{n-1} + h\left(\frac{1}{2}f_n - \frac{3}{2}f_{n-1}\right).$$

(a) What is its order?

(b) Is it 0-stable?

(c) Is it convergent?

7. What is a Householder matrix $H \in \mathbb{R}^{n\times n}$? What is a matrix $M \in \mathbb{R}^{n\times n}$ in Hessenberg form? Explain in detail the first step of the reduction of a matrix $A \in \mathbb{R}^{n\times n}$ to Hessenberg form.

8. To find the eigenvalues of a matrix $A \in \mathbb{R}^{n\times n}$ write down the QR algorithm with single shift to find the eigenvalues of matrix A. Prove that the iterates of the QR algorithm with single shift are similar, i.e., that they have the same eigenvalues.