Name:__________________________________

Ph.D. Qualifying Exam in PDE

January 2025

Note to students: This booklet contains five problems with equal weights on seven sheets of paper including this cover and the extra blank sheet on the back.

Please solve any four from these five problems in this PDE part of the exam.

You can use the back of the sheets and the extra blank page at the back.

Notes to Students and Proctor: Students are only allowed to bring pens, pencils and erasers. Backpacks and any electronic devices are strictly prohibited.

Any unauthorized item or wrong behavior during the exam will be handled according to the guidelines of the department and that of the Graduate College.

Additinal Note to Proctor: Please print the exam one-sided on seven pages of paper, extra blank papers can be provided upon requests.

1. a. Define Sobolev Space $H^1(B_1)$ where B_1 is the unit ball in the $n-$dimensional Euclidean space; And define the weak derivatives for functions in $H^1(B_1)$.

   b. State Sobolev or Morrey embedding theorems for $H^1(B_1)$ for dimension $n = 1$ and $n = 3$; and write down the these embedding theorems as inequalities for functions in $H^1(B_1)$.

2. a. Suppose u is a harmonic function defined in the domain $B_1^C = \{(x, y) : x^2 + y^2 > 1\}$, the exterior of the unit disk, and suppose that u is C^2 in B_1^C and is continuous in the closure of B_1^C. Given that $|u| \leq 1$ in the domain and that $u = 0$ on ∂B_1. Prove that $u \equiv 0$.

   b. Is the above conclusion still true in 3 dimensions? Prove your statement in this case.

3. Let $g \in L^2(\partial B_1)$ where B_1 is the unit ball in the $n-$dimensional Euclidean space. Prove the map $u \to \int_{\partial B_1} ug d\sigma$ defines a continuous linear functional on $H^1(B_1)$.

4. Let B_1 be the unit ball in $\mathbf{R}^n$. For $f \in L^2(B_1)$ and $g \in L^2(\partial B_1)$, define and prove there is a unique weak solution to boundary value problem

$$\begin{cases} -\Delta u &= f, \quad \text{in } B_1, \\ \frac{\partial u}{\partial \nu} + 2u &= g, \quad \text{on } \partial B_1. \end{cases}$$

5. Consider the heat equation on the whole x-axis. Show that there is a unique solution $u = u(x, t)$ with $0 \leq u(x, t) \leq e^{x+t}$ to

$$\begin{cases} u_t - u_{xx} &= 0, \quad x \in \mathbb{R}, \quad 0 < t < \infty, \\ u(x, 0) &= (e^x - 1)^+, \text{(the plus part)}. \end{cases}$$