Department of Mathematics University of Iowa Iowa City, IA 52242

113 MLH 9:00 am--12:00 pm, Tuesday, January 13, 2026

Ph.D. Qualifying Exam and M.S. Comp. Exam on Numerical Analysis

Note: There are two parts. **In Part I, answer 4 out of 5 questions. In Part II, answer 4 out of 5 questions.** *For each part, if you do all 5 questions, specify which 4 questions are to be graded.* You need 80% for a Ph.D. pass, and 70% for an M.S. pass. Always show your calculations and justify your answers.

Part I

1. Fred the Physicist wants to solve an equation of the form

$$(x^3 - \alpha)\, e^x - \beta = 0,$$

where α and β are parameters that can change, but are always positive and modest in size. The desired solution is also positive. He has a numerical methods textbook that mentions the bisection, fixed-point, Newton, secant, and Brent's methods. Explain which of these methods would be most appropriate for this problem with respect to speed of convergence, reliability, and appropriateness for this problem. Conclude with a recommendation of which method to use.

The next day, he says there is another equation he wants to solve, or rather a *system* of equations to be solved (number of variables equals the number of equations). Specifically, he wants to solve

$$\alpha x^3 - 2x^2 + \beta y - \gamma = 0$$
$$e^y + y - 2x^2 - xy - \theta x = 0,$$

where the parameters $\alpha, \beta, \gamma, \theta > 0$ all have modest size. Revise your recommendation in this case.

2. Explain in detail how to use Lagrange interpolation polynomials to compute the polynomial interpolant of degree $\leq n$ of (x_i, y_i) for $i = 0, 1, 2, \ldots, n$. Explain in detail how to use divided differences to compute the polynomial interpolant of the same data. Give a formula for the difference between an interpolated function f (where $y_i = f(x_i)$) and the polynomial interpolant of degree $\leq n$ that uses the $(n + 1)$th derivative of f.

3. Let $P\colon C[a,b] \to C[a,b]$ be an interpolation operator

$$Pf(x) = \sum_{i=1}^{n} f(x_i)\,\ell_i(x),$$

where $\ell_i(x_j) = 1$ if $i = j$ and zero otherwise. Also, $\|g\|_\infty = \max_{x\in[a,b]} |g(x)|$ is the max norm for continuous functions $[a,b] \to \mathbb{R}$. We define $\|P\|$ to be the smallest number β where $\|Pg\|_\infty \leq \beta \|g\|_\infty$ for all continuous g. Show that if $q \in \operatorname{range} P$, then

$$\|f - Pf\|_\infty \leq (1 + \|P\|)\|f - q\|_\infty.$$

Explain how this shows that while equally spaced polynomial interpolation suffers from the Runge phenomenon, polynomial interpolation using Chebyshev points and cubic spline interpolation (using not-a-knot cubic splines) are much more reliable. Note that for equally spaced interpolation, $\|P\| = \mathcal{O}(2^n/(n\ \ln n))$; for polynomial interpolation using Chebyshev points, $\|P\| = \mathcal{O}(\ln n)$; and for cubic spline (not-a-knot) interpolation, $\|P\| = \mathcal{O}(1)$, as $n \to \infty$.

4. Show that the basic Simpson's rule

$$\int_a^b f(x)\,dx \approx \frac{b-a}{6}\left[f(a) + 4\,f(\frac{1}{2}(a+b)) + f(b)\right],$$

is exact whenever f is a cubic polynomial. (It is sufficient to show this for $[a,b] = [-1,1]$, and then use scaling and shifting to show that it follows for any $[a,b]$). Use this or some other means to show that the composite Simpson's rule with even n,

$$\begin{aligned}\int_a^b f(x)\,dx \approx \frac{b-a}{3n}\,[f(x_0) &+ 4\,f(x_1) + 2\,f(x_2) + 4\,f(x_3) + 2\,f(x_4) + \cdots \\ &+ 2\,f(x_{n-2}) + 4\,f(x_{n-1}) + f(x_n))]\end{aligned}$$

with $x_i = a + i(b-a)/n$, has error $\mathcal{O}(n^{-4})$ as $n \to \infty$ for smooth f.

5. The degree three Legendre polynomial is $P_3(x) = \frac{1}{2}(5x^3 - 3x)$, which is degree 3 orthogonal polynomial (within a scale factor) for the inner product

$$(f,g) = \int_{-1}^{+1} f(x)\,g(x)\,dx.$$

Using this information, give the Gauss-Legendre quadrature method for computing $\int_0^1 h(t)\,dt$ of maximal order with three function evaluations. What is the maximal value of d where this method is exact for all polynomials of degree $\leq d$?

Part II

1. Apply the Gaussian elimination method with partial pivoting to solve the linear system

$$\begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

   You must provide every step that leads to your solution of the linear system.

2. Find the least squares solution of the following over-determined system

$$\begin{aligned} 2\,x_1 - x_2 &= 1, \\ x_1 + 2\,x_2 &= 1, \\ -x_1 + x_2 &= 1. \end{aligned}$$

   Show all the steps of your calculations.

3. Apply the theory of Gershgorin circles to identify three circles such that each circle contains exactly one of the three eigenvalues (let's label them λ_1, λ_2 and λ_3) of the matrix

$$A = \begin{pmatrix} 2 & 1.2 & 0 \\ 1 & -1 & 2 \\ -0.5 & 0 & 6 \end{pmatrix}.$$

   Starting with an initial vector, the power method will generate a sequence of approximate eigenvalues. What is the limit of the sequence of approximate eigenvalues for most of the choices of the initial vector?

4. Consider solving the initial value problem of a first order ordinary differential equation. What is the A-stability of a numerical method? Verify if the Euler method is A-stable. Verify if the backward Euler method is A-stable.

5. Consider the initial value problem

$$\begin{aligned} & y'' + (y')^2 + \sin(t\,y) = 1 + t, \quad t \geq 0, \\ & y(0) = 1,\ y'(0) = 0. \end{aligned}$$

   Convert the problem to one for a system of first-order ordinary differential equations. Develop a numerical method to solve the initial value problem of the system.