Ph.D. Qualifying Exam in Topology

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Instructions.

- Do eight problems: four from part A and four from part B.
- This is a closed book examination: you should have no books, technology or paper of your own. Paper will be provided by the test center.
- Please do your work on the paper provided according to the format outlined below.
  - On each page of your solutions
    - Write your name
    - Write the page number
    - Indicate which problem is being addressed
  - When you start a new problem, start a new page
  - Only write on one side of the paper
  - Make a cover page and indicate which eight problems you want graded.
- Always justify your answers unless explicitly instructed otherwise.
- You may use theorems if the problem is not a step in proving that theorem. You must to state any theorems that you use clearly and carefully.

Part A - Algebraic Topology

In the problems below, the symbols for the disk D^n, the sphere S^n and the simplex Δ^n can be understood to mean the subspaces below

$$D^n := \{v \in \mathbb{R}^n : |v| \le 1\} \qquad S^n := \{v \in \mathbb{R}^{n+1} : |v| = 1\}$$
$$\Delta^n := \{(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} : \sum_{i=0}^{n} x_i = 1\}$$

1. Fix a point $pt \in S^1$. Prove that for each group homomorphism $f : \pi_1(S^1, pt) \to \pi_1(S^1, pt)$, there exists a continuous map $g : S^1 \to S^1$ such that $f = g_*$. (Here $g_* : \pi_1(S^1, pt) \to \pi_1(S^1, pt)$ is the map induced between fundamental groups by the map g by functoriality.)

2. Use van Kampen theorem to compute $\pi_1(X_n, x_0)$ for each $n \in \mathbb{Z}_{\ge 0}$ where X_n is the one point compactification of a union of n-disjoint parallel planes through a perpendicular line:
$$X_n := \left(L \cup \left(\cup_{i=0}^{n-1} t^n(P)\right)\right)^+$$
where $P = \{(x, y, 0) : (x, y) \in \mathbb{R}^2\}$ is the xy-plane, $t : \mathbb{R}^3 \to \mathbb{R}^3$ is the map $t(x, y, z) := (x, y, z + 1)$, $L = \{(0, 0, z) : z \in \mathbb{R}\}$ and $X \mapsto X^+$ is the one-point compactification.

3. Recall that every group G has a commutator subgroup $[G, G] := \{ghg^{-1}h^{-1} : g, h \in G\}$. For each $n \ge 1$, construct the cover $p_n : Y_n \to \vee_{i=1}^n S^1$ associated to the commutator subgroup of $\pi_1(\vee_{i=1}^n S^1)$.

4. Use covering space theory to enumerate index 2 subgroups $H \subset F_3$ of the free group $F_3 := \mathbb{Z}^{*3}$ on three generators. Include illustrations of the covering spaces involved and indicate which subgroups are normal.

5. Let (C, d) be a chain complex of F-vector spaces for some field F:
$$\cdots \to C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} C_{n-2} \to \cdots$$
such that $d_{n-1}d_n = 0$ for $n \in \mathbb{Z}$. If we assume $C_n = 0$ for all but finitely many $n \in \mathbb{Z}$ then the Euler characteristic $\chi(C) \in \mathbb{Z}$ is the integer given by
$$\chi(C) := \sum_{i \in \mathbb{Z}} (-1)^i \dim_F C_i.$$

Prove that the Euler characteristic is equal to the alternating sum of dimensions of homology groups:

$$\chi(C) = \sum_{i \in \mathbb{Z}} (-1)^i \dim_F H_i(C, d)$$

6. Let $P^- := \{(x, y, z) \in \mathbb{R}^3 : z = 0 \text{ and } (x, y) \neq (0, 0)\}$ be the punctured plane and $S^2 := \{v \in \mathbb{R}^3 : |v| = 1\}$ the 2-sphere of unit radius. Compute $H_1(E)$ where $E := P^- \cup S^2$ and prove that there is no retraction of E onto P^-.

Part B - Manifolds and vector bundles

1. Give an example (you may just draw it and explain the picture) of a one-to-one immersion $f\colon M \longrightarrow N$ which is not an embedding. If M is compact, show that a one-to-one immersion $f\colon M \longrightarrow N$ is an embedding.

2. Prove that $\mathrm{U}(n) = \{A \in M_{n\times n}(\mathbb{C}) : \overline{A}^T A = \mathrm{Id}\}$ is a manifold and find its dimension.

3. Suppose $\gamma : S^1 \to \mathbb{R}^2$ is a smooth embedding. What are the possible values of the integral $\int_\gamma \eta$ if

$$\eta = \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy.$$

4. Consider the vector fields

$$X = f(x)\frac{\partial}{\partial y} \quad \text{and} \quad Y = g(y)\frac{\partial}{\partial x}$$

on $\mathbb{R}^2$. Find necessary and sufficient conditions on the pair (f, g) such that there is a diffeomorphism $h\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ with

$$h_*(\frac{\partial}{\partial x}) = X \quad \text{and} \quad h_*(\frac{\partial}{\partial y}) = Y.$$

5. Let ω_m denote the volume form of the induced metric on S^m defined by embedding S^m as the unit sphere in $\mathbb{R}^{m+1}$. Show that on each open hemisphere $x_0 \neq 0$, this volume form coincides with the restriction of

$$\frac{1}{x_0} dx_1 \wedge \ldots \wedge dx_m$$

to S^m.

6. Calculate the de Rham cohomology groups of S^2 (You may use Mayer-Vietoris)