

Ph.D. Qualifying Exam in Algebra

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Instructions:

- Do EXACTLY TWO problems from EACH of the four sections.
- Please start a new page for every new problem and put your name on each sheet.
- Please ONLY WRITE on ONE SIDE of each sheet.
- Justify your answers and show your work.
- Please write legibly.
- In answering any part of a question, you may assume the results in previous parts of the SAME question, even if you have not solved them.
- Please turn in the exam questions with your solutions.

Notations:

We adopt standard notations. Namely:

- We write \mathbb{C}, \mathbb{R} and \mathbb{Q} to denote the field of complex numbers, real numbers and rational numbers, respectively; we write \mathbb{Z} to denote the ring of rational integers; when p is a prime number, we write \mathbb{F}_p to denote the finite field with p elements.
- Throughout this exam, R denotes a ring with identity 1; R is called an integral domain if $1 \neq 0$ and R is commutative with no zero divisors.
- All R -modules are assumed to be unital left R -modules.

1 Groups

1. (15 points) Assume K is a *finite* cyclic group, H is an arbitrary group, and φ_1 and φ_2 are homomorphisms from K into $\text{Aut}(H)$ such that $\varphi_1(K)$ and $\varphi_2(K)$ are conjugate subgroups of $\text{Aut}(H)$.

Construct an explicit isomorphism to prove that $H \rtimes_{\varphi_1} K \cong H \rtimes_{\varphi_2} K$.

2. Let G be a group that acts *transitively* on a *finite* set A . (G is NOT assumed to be finite.) Let H be a normal subgroup of G , and let $\mathcal{O}_1, \dots, \mathcal{O}_r$ be the distinct orbits of H in A .

- (a) (8 points) Prove that the action of G on A induces a well-defined action of G on $\{\mathcal{O}_1, \dots, \mathcal{O}_r\}$, and prove that this action is transitive.

- (b) (7 points) Prove that all orbits of H in A have the same cardinality.

3. (15 points) Let G be a group of order 24. Prove that either

- G has a normal subgroup of order 8, or
- G has a normal subgroup N of order 4 such that G/N is isomorphic to a symmetric group S_3 .

2 Rings

1. Let \mathcal{R} be a commutative ring with $1 \neq 0$ and \mathcal{J} a proper ideal. Define a \mathcal{J} -Cauchy sequence $\{x_n\}_{n=1}^{\infty}$ to be a sequence of elements of \mathcal{R} satisfying the following condition: Given a positive integer k there exists N such that for all $m, n \geq N$ we have $x_m - x_n \in \mathcal{J}^k$.

Define addition and multiplication of sequences componentwise. A sequence $\{x_n\}_{n=1}^{\infty}$ is called a null sequence if for any positive integer k there exists N such that for all $n \geq N$ we have $x_n \in \mathcal{J}^k$.

- (a) (5 points) Prove that all \mathcal{J} -Cauchy sequences form a ring \mathcal{C} with 1.
 - (b) (5 points) Prove that all null sequences form an ideal \mathcal{N} of \mathcal{C} .
 - (c) (5 points) Assume that \mathcal{R} is a field. Prove that \mathcal{N} is a maximal ideal of \mathcal{C} .
2. Let d be a square-free *even* integer $d > 3$, and let

$$R = \mathbb{Z}[\sqrt{-d}] = \{a + b\sqrt{-d} \mid a, b \in \mathbb{Z}\}.$$

- (a) (6 points) Prove that R is an integral domain and that $R^\times = \{\pm 1\}$.
 - (b) (9 points) Prove that 2 and $\pm\sqrt{-d}$ are irreducible in R , and that 2 is not a prime element in R . Conclude that R is not a UFD.
3. Let R be an integral domain, and let D be a multiplicative subset of R such that $0 \notin D$ and $1 \in D$. Let $D^{-1}R$ be the ring of fractions of R with respect to D , and let φ be the injective ring homomorphism

$$\begin{aligned} \varphi: R &\rightarrow D^{-1}R \\ r &\mapsto r/1. \end{aligned}$$

- (a) (8 points) If J is an ideal of $D^{-1}R$, define $I_J := \varphi^{-1}(J)$. Prove that the map $J \mapsto I_J$ is an inclusion preserving injective map from the ideals of $D^{-1}R$ to the ideals of R .
- (b) (7 points) Let M be a maximal ideal of R with $D \cap M = \emptyset$. Define J_M to be the ideal of $D^{-1}R$ generated by $\varphi(M)$. Prove that $R/M \cong D^{-1}R/J_M$ as fields.

3 Linear Algebra and Module Theory

1. Let R be a ring with 1, and let A , B and C be left R -modules. Suppose $f : A \rightarrow B$ and $g : A \rightarrow C$ are R -module homomorphisms.

- (a) (7 points) Prove that the quotient D of $B \oplus C$ by $\{(f(a), -g(a)) \mid a \in A\}$ is a left R -module and that there is a commutative diagram of left R -modules

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ g \downarrow & & \downarrow \pi_1 \\ C & \xrightarrow{\pi_2} & D \end{array}$$

where π_1 and π_2 are the natural maps to the quotient D induced by the maps into the first and second components.

- (b) (8 points) Prove that if f is injective then π_2 is injective, and if f is surjective then π_2 is surjective.
2. (15 points) Let R be an integral domain, and let M be an R -module. Recall that the rank of M is the maximum number of R -linearly independent elements of M . Let n be a positive integer.

Prove: M has rank n if and only if M contains a submodule N with $N \cong R^n$ such that M/N is a torsion R -module.

3. Let V be a vector space over a field F with basis \mathcal{B} , and let $V^* = \text{Hom}_F(V, F)$ be its dual vector space. For $b \in \mathcal{B}$, let $b^* : V \rightarrow F$ be such that for all $c \in \mathcal{B}$,

$$b^*(c) = \begin{cases} 1 & \text{if } b = c, \\ 0 & \text{else.} \end{cases}$$

Define $\mathcal{B}^* = \{b^* \mid b \in \mathcal{B}\}$.

- (a) (5 points) If \mathcal{B} is finite, prove that \mathcal{B}^* is a basis of V^* .
- (b) (5 points) If \mathcal{B} is infinite, prove that \mathcal{B}^* does not span V^* .
- (c) (5 points) If \mathcal{B} is infinite, prove that V^* is isomorphic to the direct product of copies of F indexed by \mathcal{B} .

4 Field Theory and Galois Theory

- (15 points) Let F be a field, and let $F(x)$ be the field of rational functions in x with coefficients from F . Let $y = \frac{P(x)}{Q(x)}$ be a rational function, with relatively prime polynomials $P(x), Q(x) \in F[x]$, $Q(x) \neq 0$. Define $n = \max(\deg P(x), \deg Q(x))$, and suppose $n \geq 1$.

Let $F(y)$ be the minimal subfield of $F(x)$ containing F and y . Prove that

$$[F(x) : F(y)] = n.$$

- (15 points) Let $f(x) = x^5 - 6x + 3 \in \mathbb{Q}[x]$. Prove that $f(x)$ cannot be solved by radicals over \mathbb{Q} .
- Let p be a prime number, let $a \in \mathbb{F}_p^\times$, and let $f(x) = x^p - x + a \in \mathbb{F}_p[x]$.
 - (8 points) Prove that $f(x)$ is irreducible and separable over \mathbb{F}_p .
 - (7 points) Determine the Galois group G of $f(x)$ over \mathbb{F}_p . Moreover, find generators of G and determine how these generators act on the roots of $f(x)$.