## Ph.D. qual. exam. and M.S. comp. exam. on ODEs.
## Wednesday January 14, 2026.

Answer 5 out of 9 questions (only the best 5 will count). Show your calculations and justify your answers.

1. The function
$$x(t) = x(t, t_0, x_0)\Big|_{t_0=0, x_0=0} = \sin(t)$$
is solution to the following nonautonomous ODE
$$\dot{x} = (\sin(t) - 3)(x - \sin(t)) + \cos(t)$$
with initial conditions $x_0 = 0$ at $t_0 = 0$, i.e., $x(0) = 0$. Give explicitly the (sensitivity) function
$$R(t, 0, 0) := D_{x_0} x(t, t_0, x_0)\Big|_{t_0=0, x_0=0} = \frac{\partial}{\partial x_0} x(t, t_0, x_0)\Big|_{t_0=0, x_0=0} \quad .$$

2. Consider the flow $\varphi_t$ of the following autonomous system of differential equations
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} x_2 - x_1^2 - x_1 - x_2 x_3 \\ x_1 x_2 + x_1 x_3 + x_2 \\ -2x_1 x_2 + x_1 x_3 - x_3 \end{bmatrix}$$
and the set of initial values at $t_0 = 0$
$$X_0 := \{x = (x_{01}, x_{02}, x_{03}) \in \mathbb{R}^3 \mid -1 \leq x_{0i} \leq 1 \text{ for } i = 1, 2, 3\}.$$
What is the volume of the set $\varphi_t(X_0)$ at $t = 2$, i.e., mathematically speaking what is the value of $\mathrm{Vol}(\varphi_2(X_0))$?

3. We consider the following system of nonlinear autonomous ODEs
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_1(2 - x_1 - x_2) \\ x_2(2 - 4x_1 - x_2) \end{bmatrix}$$
Find all fixed points. Which of these fixed points are hyperbolic? If a fixed point is hyperbolic say if it is a source, a sink, or a saddle.

4. The origin $x^* = (0, 0, 0)$ is a fixed point of the autonomous system of differential equations
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -2x_2 + x_2 x_3 - x_1^3 \\ x_1 - x_1 x_3 - x_2^3 \\ x_1 x_2 - x_3^3 \end{bmatrix}.$$
   (a) Is the origin $x^* = (0, 0, 0)$ a hyperbolic fixed point?
   (b) Is the function $V(x_1, x_2, x_3) := x_1^2 + 2x_2^2 + x_3^2$ a (strict) Lyapunov function at the origin $x^* = (0, 0, 0)$ for this system of differential equations?
   (c) Is the constant solution $x(t) = x^* = (0, 0, 0)$ asymptotically Lyapunov-stable?

5. (a) Consider the system of ODEs given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 - 7x_1\left(x_1^2 + x_2^2\right) \\ x_1 - x_2 - 7x_2\left(x_1^2 + x_2^2\right) \end{bmatrix}.$$

Does this system have a periodic orbit?

(b) Consider the system of ODEs given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 - 7x_1\left(x_1^2 + x_2^2\right) \\ x_1 + x_2 - 7x_2\left(x_1^2 + x_2^2\right) \end{bmatrix}.$$

Does this system have a periodic orbit?

Note: a fixed point is not considered as being a periodic solution.

6. Consider the autonomous nonlinear system of ODEs

$$\dot{x}_1 = 2x_1x_2 + x_2, \quad \dot{x}_2 = x_1^2 - 3x_2^2 + x_1.$$

Does this system have a periodic orbit? (note: a fixed point is not considered as being a periodic orbit.)

7. Does there exist a function $y \in \mathcal{C}^1([0,1],\mathbb{R})$ such that $y(0) = 1/2$, $y(1) = 1$ and the first variation of

$$\mathcal{A}[y] := \int_0^1 y^2 + (x+y)y'dx = \int_0^1 y^2(x) + (x + y(x))y'(x)dx$$

satisfies $\delta\mathcal{A}[y] \equiv 0$, i.e., $\delta\mathcal{A}[y][h] = 0 \ \forall h \in \mathcal{C}_0^1([0,1],\mathbb{R})$?

8. For the following one-dimensional autonomous nonlinear ODE

$$\dot{x} = x^3 + x^2 - \mu x - \mu = (x+1)(x^2 - \mu),$$

find all the fixed points as a function of the parameter $\mu \in \mathbb{R}$, determine their Lyapunov-stability, sketch a bifurcation diagram, and find the bifurcation value(s) of the parameter $\mu \in \mathbb{R}$. Classify the bifurcation(s) as of the type *saddle-node*, *transcritical*, or *pitchfork*.

9. Consider the system of ODEs $\dot{x} = f(t,x)$ and the following explicit Runge-Kutta method

$$\begin{aligned} X_1 &= x_0 \\ X_2 &= x_0 + h\frac{1}{2}f(t_0, X_1) \\ X_3 &= x_0 + h\left(-f(t_0, X_1) + 2f(t_0 + h/2, X_2)\right) \\ x_1 &= x_0 + h\left(\frac{1}{6}f(t_0, X_1) + \frac{2}{3}f(t_0 + h/2, X_2) + \frac{1}{6}f(t_0 + h, X_3)\right) \end{aligned}$$

(a) What is the (local) order of this method?

(b) What is the stability function $R(z)$ of this method ($\dot{x} = \lambda x$, $z := h\lambda$, and $x_1 = R(z)x_0$)?