Name:__________________________________

Ph.D. Qualifying Exam in PDE

January 2026

Note to students: This booklet contains four problems with equal weights on six sheets of paper including this cover and the extra blank sheet on the back.

You can use the back of the sheets and the extra blank page at the back.

Notes to Students and Proctor: Students are only allowed to bring pens, pencils and erasers. Backpacks and any electronic devices are strictly prohibited.

Any unauthorized item or wrong behavior during the exam will be handled according to the guidelines of the department and that of the Graduate College.

Additinal Note to Proctor: This is PDE part of the Differential Equation qualifying exam, which consists of ODE and PDE parts. Please print the exam one-sided on six pages of paper, extra blank papers can be provided upon requests.

1. a. Define weak derivatives and Sobolev Space $W^{1,p}(B_1)$ where $1 \le p \le \infty$ where B_1 is the unit ball in the $n-$dimensional Euclidean space.

   b. Find all p so that $\frac{1}{r} \in W^{1,p}(B_1)$? Here B_1 the three dimensional unit ball and $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$.

   c. State Sobolev or Morrey embedding theorems for $W^{1,4}(B_1)$ for dimension $n = 1$ and $n = 3$ respectively; and write down these embedding theorems as inequalities for functions in $W^{1,4}(B_1)$.

2. a. Suppose u is a smooth harmonic function defined in $\mathbf{R}^n$. Also suppose that $|u(x)| \le 2026\sqrt{1 + |x|}$.

   Prove u is a constant and $|u| \le 2026$.

3. Let B_1 is the unit ball in $\mathbf{R}^n$. For $f \in L^2(B_1)$ and $g \in L^2(\partial B_1)$, define and prove there is a unique weak solution to boundary value problem

$$\begin{cases} -\Delta u &= f, \quad \text{in } B_1 \\ \frac{\partial u}{\partial \nu} + 2u &= g, \quad \text{on } \partial B_1. \end{cases}$$

   Can you relax the condition $g \in L^2(\partial B_1)$?

4. Consider the heat equation on the real-axis

$$\begin{cases} u_t - u_{xx} &= 0, \quad x \in \mathbb{R}, \quad 0 < t < \infty, \\ u(x,0) &= \sin(x). \end{cases}$$

   Show there is a unique bounded solution.