Ph.D. Qualifying Exam and M.S. Comprehensive Exam in Algebra

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Instructions:

- Do EXACTLY TWO problems from EACH of the four sections.
- Please start a new page for every new problem and put your name on each sheet.
- Justify your answers and show your work.
- Please write legibly.
- In answering any part of a question, you may assume the results in previous parts of the SAME question, even if you have not solved them.
- Please turn in the exam questions with your solutions.

Notations:

We adopt standard notations. Namely:

- We write $\mathbb{C}, \mathbb{R}$ and $\mathbb{Q}$ to denote the field of complex numbers, real numbers and rational numbers, respectively. We write $\mathbb{Z}$ to denote the ring of rational integers. If p is a prime number and $q = p^k$ $(k \geq 1)$, then $\mathbb{F}_q$ denotes the finite field with q elements.
- Throughout this exam, R denotes a ring with identity $1 \neq 0$; R is called an integral domain if it is commutative with no zero divisors.
- All R-modules are assumed to be unital left R-modules.

1 Groups

1. Let G be an *arbitrary* group and assume it acts on the set of vertices of a pentagon. Show that G has a normal subgroup of finite index at most 200.

2. Let G be a group with 154 elements which is indecomposable, that is, G is not isomorphic to a product of groups $H \times L$ for groups H, L of smaller order. Show that G is isomorphic to the dihedral group D_{154} with 154 elements.

3. Let $p \in \mathbb{Z}$ be a prime number and $F = \mathbb{Z}/p\mathbb{Z}$ the field with p elements. Consider the group G of invertible, upper-triangular 4×4 matrices with entries in F. Prove that G has a unique Sylow p-subgroup and find its order.

2 Rings

1. Let $w = 5 - 2i \in \mathbb{Z}[i]$.

   (i) Show that w is an irreducible element of the ring $\mathbb{Z}[i]$. Is w a prime element of this ring?
   (ii) Show that $F = \mathbb{Z}[i]/\langle w \rangle$ is a finite field with 841 elements.

2. Show that there is an isomorphism of rings

$$\frac{\mathbb{R}[X]}{\langle X^{12} - 1 \rangle} \cong \mathbb{R}^k \times \mathbb{C}^s$$

   and determine the numbers k, s.

3. Let R be a principal ideal domain (PID) and F the field of fractions of R. Identify R as the subring of F consisting of fractions $r/1$ where $r \in R$. Now suppose we have another subring $R \subseteq S \subseteq F$.

   (i) Prove that every element of S is of the form a/b with $a, b \in R$ and $1/b \in S$.
   (ii) Prove that S is a PID.

3 Linear Algebra and Module Theory

1. Let $A \in M_n(\mathbb{C})$ be a matrix which has distinct eigenvalues $\lambda_1, \ldots, \lambda_n$. Assume that the set $\{\lambda_i^k | 1 \le i \le n, k \ge 1\}$ is finite. Show that the set $\{A^k | k \ge 1\}$ is also finite.

2. Let $A \in M_{2,3}(\mathbb{C})$ and $B \in M_{3,2}(\mathbb{C})$ be matrices such that $BA = \begin{pmatrix} 2 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 2 & 0 \end{pmatrix}$.

   If AB is invertible, find $\det(AB)$.

3. Let $\phi\colon \mathbb{Z}^4 \to \mathbb{Z}^4$ be a group homomorphism such that:

$$\phi(1,0,0,0) = (1,3,6,5)$$

$$\phi(0,1,0,0) = (4,-3,1,1)$$

$$\phi(0,0,1,0) = (-2,0,1,0)$$

$$\phi(0,0,0,1) = (3,3,3,3)$$

   Identify the isomorphism class of the $\mathbb{Z}$-module $\operatorname{coker}(\phi)$.

4 Field Theory

1. Let ζ be a primitive 3rd root of unity and $\sqrt[3]{2}$ the real 3rd root of 2. Prove that $K = \mathbb{Q}(\zeta, \sqrt[3]{2})$ is a Galois extension of $\mathbb{Q}$ and describe its Galois group.

2. Let K be a field and $f(x) \in K[x]$ irreducible, separable, and of odd degree. Let G be the Galois group of $f(x)$. Show that G does not contain a central element of order 2.

3. Let $f(x) \in \mathbb{Q}[x]$ be a polynomial of degree 9, and $F \supset \mathbb{Q}$ its splitting field. Prove that there is no subfield of F of degree 25 over $\mathbb{Q}$.