Ph.D. Qualifying Examination in Analysis

Professors Ionut Chifan and Paul Muhly

January 9, 2019

Instructions. Be sure to put your name on each booklet you use.

This examination has a number of “true-false" questions in it. When a problem is a true-false problem, the operative statement will be preceded by **True-False?**. You are to decide whether it is true or false. If you think it is true, you must provide a proof. If you think it is false, you must provide a counter example or a proof of why it is false. No points will be given for a correct guess that the problem is true or false without any justification. Also, there will be no “Bankruptcy” points given.

The exam is divided into two parts. The first covers real analysis and the second covers complex analysis. Each part has 5 problems. You need only work 4 problems in each part. You must indicate which 4 you are submitting for evaluation. If you want to do five in a part, that is OK. We will treat the extra problem as a bonus, but still mark the four problems (in each part) that you think are your best work.

Part I

1. Let f be a real-valued continuous function mapping $[0,1]$ to $[0,1]$. **True-False?** f is absolutely continuous if and only if f maps Lebesgue null sets to Lebesgue null sets.
2. Suppose f is a non-negative Lebesgue integrable function defined on $[0,1]$. **True-False?** There is a Lebesgue measurable set $E \subseteq [0,1]$ such that $f = 1_E$ if and only if $\int_0^1 f^n \, d\mu = \int_0^1 f \, d\mu$ for all positive integers n.
3. For what values of $\alpha > 0$ is the function $x \to x^\alpha$ absolutely continuous on every bounded subinterval of $[0,\infty)$?
4. Let σ be the function defined by the formula
$$\sigma(x) := \begin{cases} 0, & x \leq 0 \\ 1, & x > 0 \end{cases}$$
and let σ^* be the outer measure determined by σ. **True-False** Every subset of $\mathbb{R}$ is measurable with respect to σ^*.
5. **True-False** The function
$$f(x) := \sum_{k=1}^{\infty} \sin(kx)/k^m$$
is of bounded variation over every finite interval whenever $m > 2$.

Part II

6. Suppose f is holomorphic in the open unit disc $\mathbb{D}$. Suppose also that for each $z \in \mathbb{D}$ there is an integer $n(z)$ such that the derivative $f^{(n(z))}$ vanishes at z. **True-False?** f must be a polynomial.
7. **True-False?** There is a holomorphic function f on the closed unit disc such that $f(\frac{1}{n}) = \frac{1}{n+2}$, $n \geq 1$.
8. A function f defined and analytic on a region G is said to have a *fixed point* z in G if $f(z) = z$. If f is analytic in a region that contains the closed unit disc and if $|f(z)| < 1$ for all z, $|z| = 1$, how many fixed points does f have in the open unit disc?
9. **True-False?** The function of r
$$\varphi(r) := \int_{|z|=r} \frac{\sin z}{z^2+1} \, dz, \quad r \neq 1$$
can be extended to a continuous function defined on all of $[0,\infty)$.

10. Recall that a function f defined on the extended complex plane $\widehat{\mathbf{C}} := \mathbb{C} \cup \{\infty\}$ is said to be meromorphic on $\widehat{\mathbf{C}}$ in case f is meromorphic on $\mathbb{C}$ and $f(\frac{1}{z})$ has a non-essential singularity at $z = 0$. Show that a nonconstant function that is meromorphic on $\widehat{\mathbb{C}}$ has the same number of zeros and poles in $\widehat{\mathbf{C}}$.