Fractions and Rational Expressions

Math Tutorial Lab Special Topic*

What is a fraction?

A fraction is a number that we can write in the form $\frac{a}{b}$, where a and b are whole numbers. The fraction bar is actually telling us to divide the two numbers – for example, $\frac{2}{3}$ is the same as $2 \div 3$. The top of the fraction is called the **numerator** and the bottom of the fraction is called the **denominator**.

Simplifying fractions

We typically want to write fractions in lowest terms – that is, we want to cancel any common factors of the numerator and the denominator. For example, consider $\frac{18}{4}$. This fraction is not in lowest terms, as we could write $\frac{18}{4} = \frac{9 \cdot 2}{2 \cdot 2}$. Both the numerator and the denominator have a factor of 2, and thus, we can cancel a factor of 2 from both the numerator and the denominator. In this fashion, we can see that $\frac{18}{4} = \frac{9}{2}$.

Example Simplify $\frac{21}{28}$.

Example Simplify $\frac{54}{13}$.

Adding and Subtracting Fractions

When adding or subtracting two fractions, we must make sure that the denominator of the fractions match. If the denominators are different, we have to find a common denominator before we add or subtract. The common denominator is going to be the least common multiple of our denominators. In other words, the common denominator is the smallest number that is divisible by both of the denominators. You want to include all factors of the denominators in your common denominator, but no extra factors. We then rewrite our fractions using the common denominator, and then add or subtract the numerators. A step-by-step

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example is given below.

Example Compute $\frac{4}{9} + \frac{11}{12}$.

- Step 1: Notice that the denominators of our fractions are not the same. We must first determine the common denominator – the least common multiple of 9 and 12. We can write out some multiples of 9 to see this: $9 \cdot 1 = 9, 9 \cdot 2 = 18, 9 \cdot 3 = 27, 9 \cdot 4 = 36$. But, 12 divides 36, so we've found our least common multiple. Our common denominator is then 36.

- Step 2: We now need to rewrite our fractions with our common denominator. That is, we want to write $\frac{4}{9} = \frac{?}{36}$. In this case, we have to multiply the denominator, 9, by 4 to get 36, so we also have to multiply the numerator by 4 as well. Therefore, we have $\frac{4}{9} = \frac{4 \cdot 4}{9 \cdot 4} = \frac{16}{36}$. Similarly, $\frac{11}{12} = \frac{11 \cdot 3}{12 \cdot 3} = \frac{33}{36}$.

- Step 3: Now, we can add our new fractions together. To do this, we simply add the numerators: $\frac{16}{36} + \frac{33}{36} = \frac{16+33}{36} = \frac{49}{36}$.

In the above example, notice that our final fraction, $\frac{49}{36}$, is already in lowest terms, and doesn't need to be simplified. But, in some cases, you may have to simplify after adding or subtracting. Although we demonstrated addition above, the process is the same for subtraction.

Example Compute $\frac{7}{8} - \frac{2}{4}$.

Example Compute $\frac{11}{6} + \frac{15}{9}$.

Multiplying Fractions

When multiplying fractions, we no longer have to worry about a common denominator. To multiply two fractions, we just multiply the numerators and multiply the denominators. Here's an example:

Example Multiply $\frac{6}{7}$ and $\frac{8}{3}$.

We have $\frac{6}{7} \cdot \frac{8}{3} = \frac{6 \cdot 8}{7 \cdot 3} = \frac{48}{21}$. Notice that the numerator and the denominator of this fraction both contain a factor of 3 that we can cancel. So, $\frac{48}{21} = \frac{16 \cdot 3}{7 \cdot 3} = \frac{16}{7}$.

In the above example, we could have simplified at any step of the multiplication. All we have to recognize is that there will be a common factor in both the numerator and the denominator, and then we can cancel. Sometimes it may be easier to try to simplify and cancel common factors before completing the multiplication.

Example Multiply $\frac{11}{12}$ and $\frac{3}{4}$.

Example Multiply $\frac{13}{8}$ and $\frac{20}{21}$.

Dividing Fractions

Dividing two fractions can actually be turned into a multiplication problem. We take the fraction we are dividing by, and take the reciprocal (flip the fraction around). Then, we multiply instead of dividing.

Example Compute $\frac{2}{5} \div \frac{7}{3}$.

We first find the reciprocal of $\frac{7}{3}$. Remember that the reciprocal just turns the fraction upside down, so the reciprocal of $\frac{7}{3}$ is $\frac{3}{7}$. Now, we multiply instead of divide: $\frac{2}{5} \div \frac{7}{3} = \frac{2}{5} \cdot \frac{3}{7} = \frac{2 \cdot 3}{5 \cdot 7} = \frac{6}{35}$.

Sometimes, we may have a fraction within a fraction. For example, $\frac{\frac{2}{5}}{\frac{7}{3}}$. But, remember that the fraction bar simply means division, so this can be written as $\frac{2}{5} \div \frac{7}{3}$, and computed as in our example above.

Example Compute $\frac{8}{3} \div \frac{5}{3}$.

Example Compute $\frac{7}{8} \div \frac{1}{2}$.

Rational Expressions

A rational expression is a fraction with polynomials in the numerator and denominator instead of whole numbers as we have seen in the examples above. Some examples of rational expressions include $\frac{x-2}{x+3}$, $\frac{1}{(x+5)^2}$, and $\frac{x^2+3x+1}{x^2-4}$. Just like regular fractions, we can simplify, add, subtract, multiply, and divide rational expressions.

Simplifying Rational Expressions

Much like the regular fractions we saw above, we can simplify rational expressions if the numerator and the denominator have a common factor. However, instead of simply having a whole number as a factor, remember that polynomials often have more complicated factors. For example, consider $x^2 + x - 6$. If we factor this polynomial, we have $x^2 + x - 6 = (x - 2)(x + 3)$. So, $x - 2$ and $x + 3$ are the factors of $x^2 + x - 6$. Thus, when simplifying rational expressions, you may need to factor the numerator and the denominator to more clearly see the common factors.

Example Simplify $\frac{x^2-3x-10}{x^2+5x+6}$

Example Simplify $\frac{4x^2+16x+16}{2x^2+6x+4}$

Adding and Subtracting Rational Expressions

When adding or subtracting rational expressions, we again need to find a common denominator. This can be a bit challenging with polynomials as our denominators, so make sure to factor as much as possible before trying to find the common denominator. To find the common denominator, you want to look at all the factors of the denominators, and then multiply the factors together. After this, we can proceed with the addition or subtraction as we did with regular fractions.

Example Compute $\frac{x-2}{x-3} + \frac{x}{x^2-2x-3}$.

- Step 1: In order to clearly see the factors of the denominators, we first factor $x^2 - 2x - 3$. We see that $x^2 - 2x - 3 = (x - 3)(x + 1)$. The other denominator, $x - 3$, is already fully factored. Therefore, the factors of our denominators are $x - 3$ and $x + 1$. We multiply these to get our common denominator: $(x - 3)(x + 1) = x^2 - 2x - 3$.
- Step 2: We now rewrite the fractions with our common denominator. The denominator of the first fraction, $x - 3$, needs to be multiplied by $x + 1$ to match the common denominator. So, we have

$\frac{x-2}{x-3} = \frac{(x-2)(x+1)}{(x-3)(x+1)} = \frac{x^2-x-2}{x^2-2x-3}$. The second fraction, $\frac{x}{x^2-2x-3}$, already has the common denominator in the denominator, so we don't need to change it.

- Step 3: We can now add the two fractions. To do this, we just add the numerators together: $\frac{x^2-x-2}{x^2-2x-3} + \frac{x}{x^2-2x-3} = \frac{x^2-x-2+x}{x^2-2x-3} = \frac{x^2-2}{x^2-2x-3}$.

Example Compute $\frac{1}{x+3} + \frac{x}{x^2+6x+9}$.

Example Compute $\frac{1}{x+5} + \frac{2}{2x-10} - \frac{2x}{x^2-25}$.

Multiplying and Dividing Rational Expressions

Multiplying and dividing with rational expressions follows the same process we saw above with regular fractions. To multiply two rational expressions, we simply multiply the numerators and the denominators. When dividing two rational expressions, we find the reciprocal of the expression we're dividing by, and instead multiply by the reciprocal.

Example Multiply $\frac{x^2+3x-28}{10x+70}$ and $\frac{-9x-9}{x^2-4x}$.

Example Compute $\frac{-9x-72}{x-7} \div \frac{x+10}{x^2+3x-70}$.