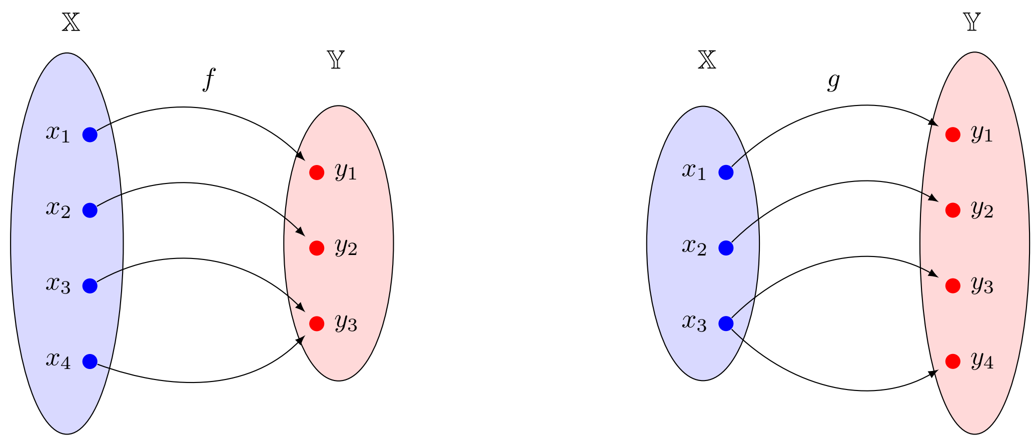
Function Review

Math Tutorial Lab Special Topic*

What is a function?

A **function** from a set $\mathbb{X}$ to a set $\mathbb{Y}$ is a rule that assigns each element in $\mathbb{X}$ to precisely one element in $\mathbb{Y}$. To illustrate, examine the functions below:



(a) A Function

(b) Not a Function

Figure 1

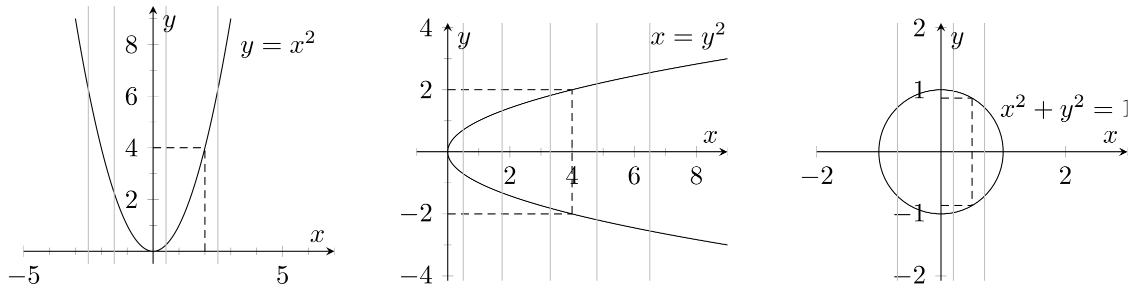
The mapping f in Figure 1a is a function, as it takes each element in $\mathbb{X}$ to one element in $\mathbb{Y}$. Notice that two elements in $\mathbb{X}$ can be mapped to the same element in $\mathbb{Y}$, as seen by both x_3 and x_4 being mapped to y_3. On the other hand, the mapping g in Figure 1b is not a function, as it sends the element x_3 to both y_3 and y_4, two different outputs in the set $\mathbb{Y}$.

We often use the **vertical line test** on a graph of an equation to determine if a given equation is in fact a function or not. Several vertical lines are drawn on the graph, and we determine how many times the vertical line crosses the graph of the equation. An equation describes y as a function of x if and only if every vertical line intersects the graph of the equation exactly once for each x. Some examples are given in Figure 2.

Every vertical line intersects the graph of $y = x^2$, as shown in Figure 2a, in exactly one place, so for each real number x, there is precisely one point $(x, y) = (x, x^2)$ on the graph. This means that the graph of $y = x^2$ is the graph of a function.

On the other hand, there are vertical lines that intersect the graph of $x = y^2$ twice. For example, the line $x = 4$ in Figure 2b intersects the graph of $x = y^2$ at both $y = 2$ and $y = -2$, so the graph of $x = y^2$ is not the graph of a function.

*Created by Maria Gommel, June 2014



(a) A function (b) Not a function (c) Not a function

Figure 2

Similarly, the dashed line $x = \frac{1}{2}$ in Figure 2c intersects the graph of $x^2 + y^2 = 1$ at both $y = \frac{\sqrt{3}}{2}$ and $y = -\frac{\sqrt{3}}{2}$, so the graph of $x^2 + y^2 = 1$ is also not the graph of a function.

Functions are often denoted by letters such as f, F, g, G and so on. We refer to $f(x)$ as the value of f at the number x. Thus, $f(x)$, read "f of x," is the number that results when x is given and the rule for f is applied. We often call $f(x)$ the "image of x under f". **Warning:** $f(x)$ does not mean "f times x."

Example If $f(x) = 2x^2 + 3$, find

(a) $f(2)$

(b) $f(2x)$

(c) $f(x + h)$

(d) $f(\text{☺})$

Domain and Range

Every function has a domain and range. The **domain** of a function is all numbers that can be input into the function, i.e. all elements, x, that can legally be used by the function, f. The **range** of a function is the set of possible outcomes (results) after inputting the elements of the domain.

Listed below is a summary of some important items to remember about a function f:

- $f(x)$ is the image of x or the value of f at x when the rule f is applied to an x in the domain.
- To each x in the domain of f, there is one and only one image $f(x)$ in the range
- f is the symbol we use to denote the function. It is symbolic of a domain and a rule we use to get from an x in the domain to $f(x)$ in the range.

Many functions can take any input (their domain is all real numbers, or $(-\infty, \infty)$). However, a few common functions have restrictions to their domains. For example, for any function with a square root, we have to exclude from the domain all numbers that make the radicand of the square root negative, as we cannot take a square root of a negative number. Another example occurs when a function contains a fraction. As fractions represent division, and we cannot divide by zero, we must exclude from the domain any numbers which cause the denominator of the fraction to be zero. Some examples illustrating these concepts are given below.

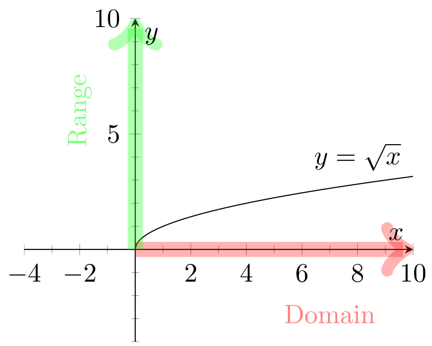


Figure 3

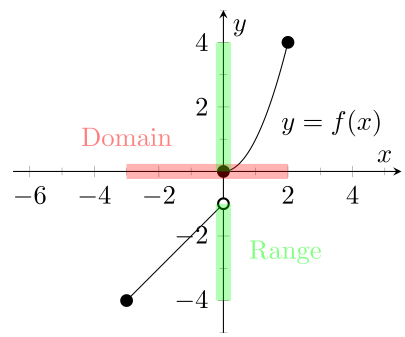


Figure 4

Example Find the domain and range of $f(x) = x^2$.

Example Find the domain and range of $f(x) = \sqrt{x}$. (Hint: See Figure 3)

Example Find the domain and range of $f(x) = \frac{1}{x}$.

A function that is defined by differing expressions on various portions of its domain is called a **piecewise-defined** function.

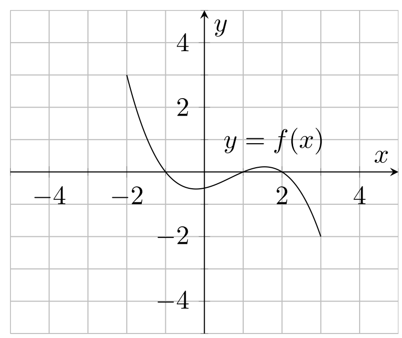
Example Sketch the graph and find the domain and range of the function defined by

$$f(x) = \begin{cases} x - 1, & \text{if} \quad -3 \leq x < 0 \\ x^2, & \text{if} \quad 0 \leq x \leq 2. \end{cases}$$

(Hint: See Figure 4)

Example Find the domain of $f(x) = \frac{1}{\sqrt{x^2-4}}$.

Example The graph of the function f is given in the figure.



(a) Determine the values $f(-1), f(0), f(1), f(3)$.

(b) Determine the domain and range of the function.

One last bit of terminology - if x is in the domain of f, we shall say that f is defined at x or that $f(x)$ exists. If x is not in the domain of f, we say that f is not defined at x or that $f(x)$ does not exist. In our example above, we would say that f is defined at 2, since 2 is in the domain of f. Alternatively, f is not defined at -3, since -3 is not in the domain of f.

Arithmetic Combinations of Functions

We can combine two functions using our well-known arithmetic operations as follows. If f and g are functions, then the functions $f+g$, $f-g$, $f \cdot g$, and f/g are defined by

$$(f+g)(x) = f(x) + g(x), \quad (f-g)(x) = f(x) - g(x),$$
$$(f \cdot g)(x) = f(x) \cdot g(x), \quad (f/g)(x) = \left(\tfrac{f}{g}\right)(x) = \tfrac{f(x)}{g(x)}.$$

Example If $f(x) = x^2$ and $g(x) = 3x - 2$, compute $f+g, f-g, f \cdot g$, and f/g.

Both $f(x)$ and $g(x)$ must be defined for any of these arithmetic operations to be defined at x. This means that the domains of $f+g$, $f-g$, and $f \cdot g$ consist of those real numbers that are common to both the domain of f and the domain of g. The domain of the quotient f/g consists of those real numbers x that are in both the domain of f and the domain of g, and that also satisfy $g(x) \neq 0$.

Composition of Functions

The **composition** of the function f with the function g, denoted $f \circ g$, is defined by $(f \circ g)(x) = f(g(x))$. The domain of $f \circ g$ consists of those x in the domain of g for which $g(x)$ is in the domain of f. This is illustrated by the diagram below:

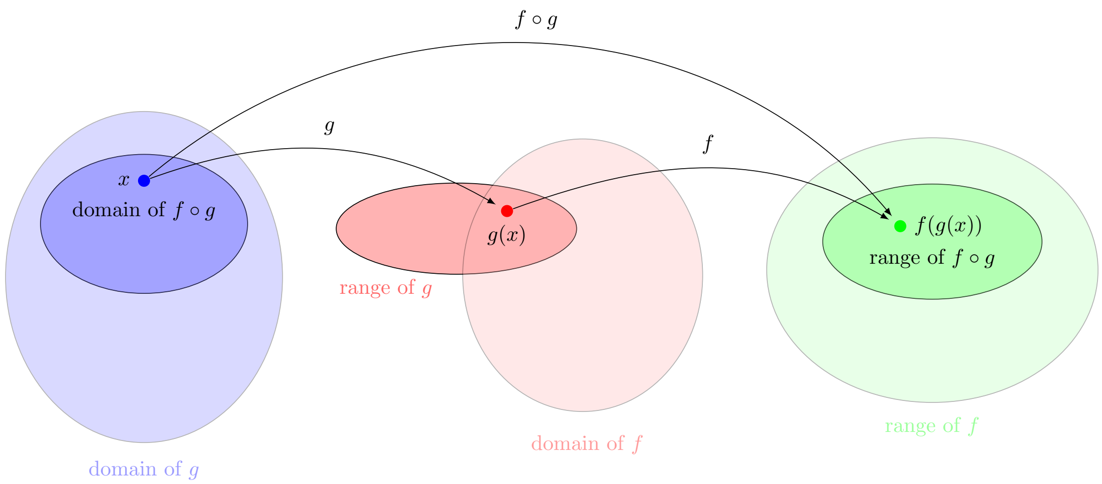


Figure 5

When computing $(f \circ g)(x) = f(g(x))$, first x is inputted into the function g, producing the output $g(x)$. Then $g(x)$ is used as the input to the function f. This gives the final value of the composition. The composition is defined provided x is a valid input for the function g, that is, x is in the domain of g, and, in addition, $g(x)$ is in the domain of f. Similarly, when computing $(g \circ f)(x) = g(f(x))$, first x is inputted to the function f, producing the output $f(x)$. Then, we use $f(x)$ as the input to the function g, which will give the final value of the composition.

Example Let $f(x) = \sqrt{x-1}$ and $g(x) = \frac{1}{x^2}$.

(a) Is $(f \circ g)(0)$ defined?

(b) Is $(f \circ g)(2)$ defined?

(c) Find $(f \circ g)(x)$.

(d) Find $(g \circ f)(x)$.

Warning: As you can see from the example above $(f \circ g)(x) \neq (g \circ f)(x)$. This is typically the case when computing compositions.

References

Most definitions, examples, and other text was taken from the fifth edition of *PreCalculus* by J. Douglas Faires and James DeFranza.

Code for Figures 1 and 5 adapted from `http://tex.stackexchange.com/questions/72719/how-to-draw-function-domain-co-domain-range-diagram?rq=1`.