LOW COMPLEXITY SUBSHIFTS

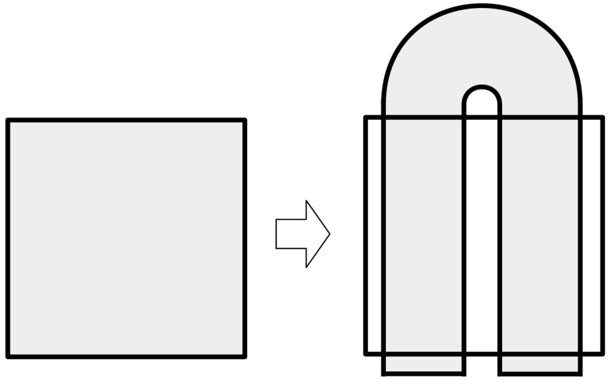
Nic Ormes, University of Denver

DYNAMICAL SYSTEMS

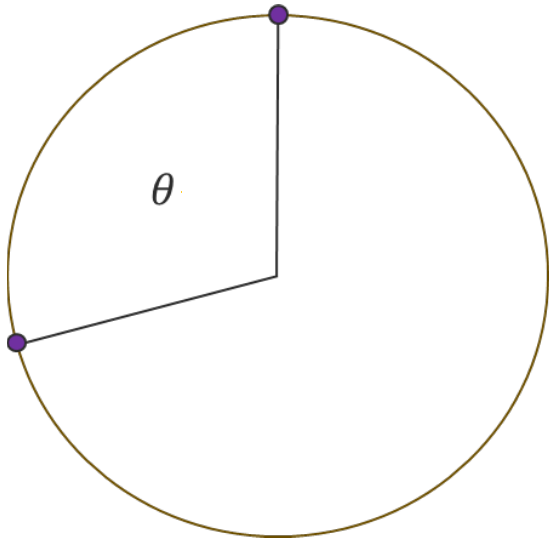
Homeomorphism of a compact metric space $T: X \to X$

Examples to think about:

Smale Horseshoe

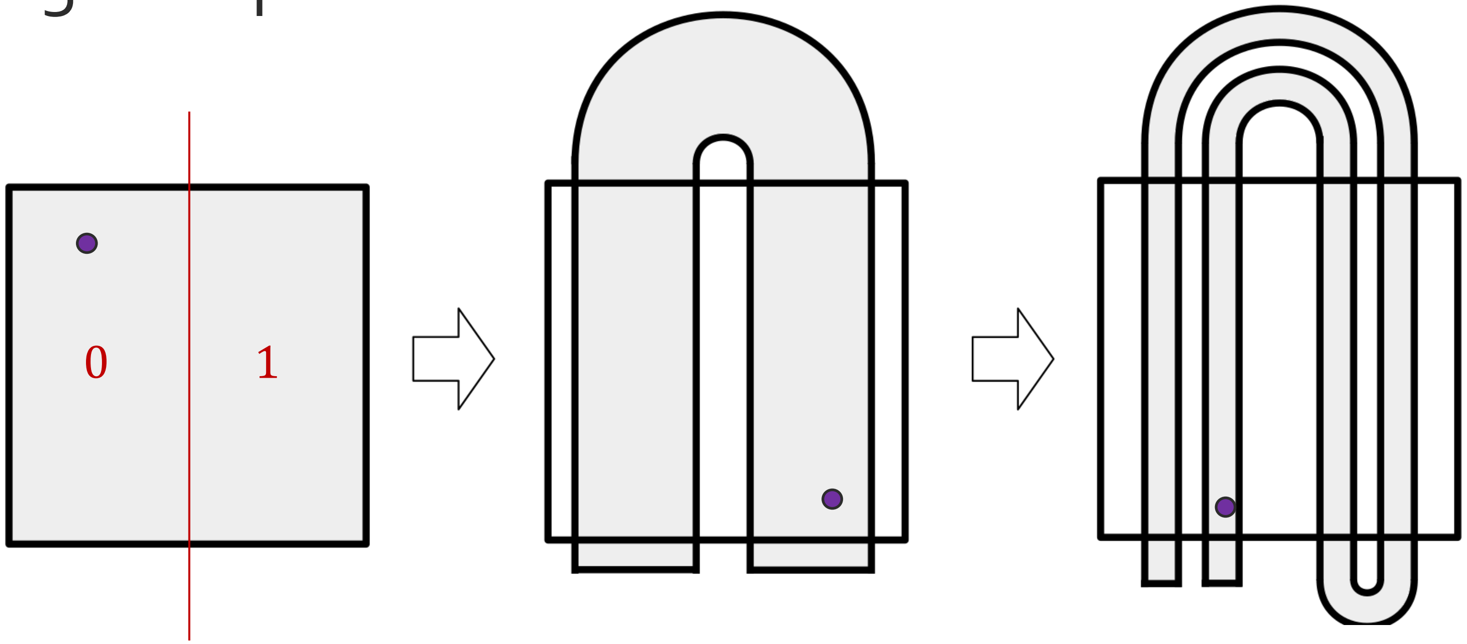


Irrational Rotation



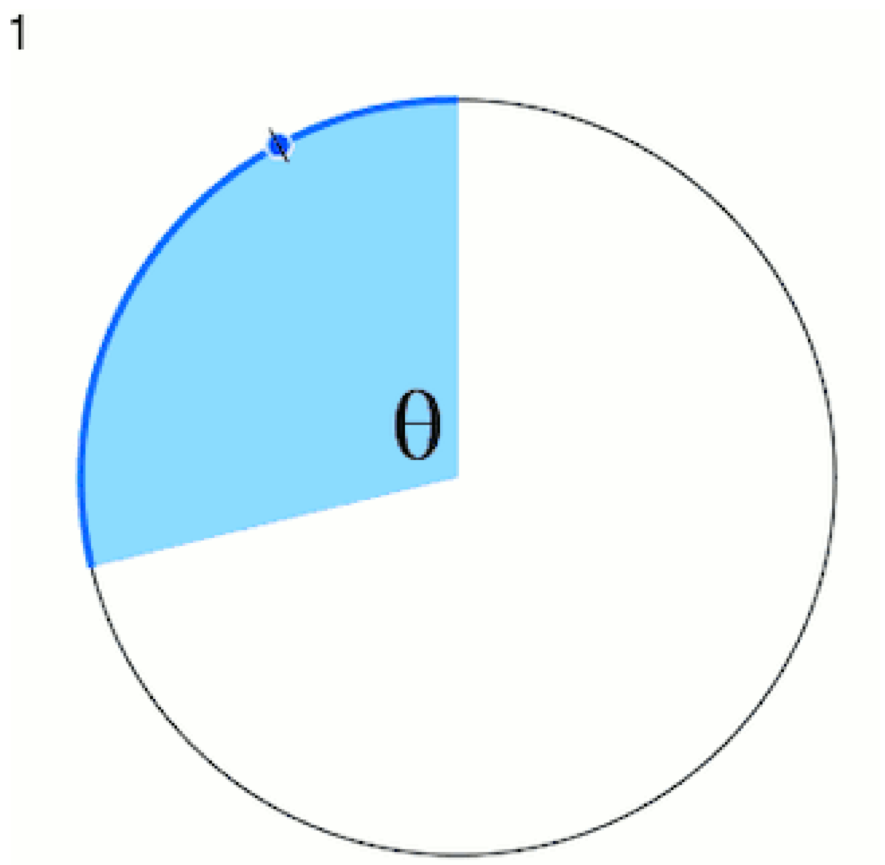
DYNAMICAL SYSTEMS → SYMBOLIC DYNAMICS

Partition the space into finitely many sets. Code every point by their itinerary through the partition.



… 0 1 0 …

DYNAMICAL SYSTEMS → SYMBOLIC DYNAMICS



SUBSHIFTS

Full shift: $\mathcal{A}$ – finite set of symbols and $\sigma: \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ is the shift map

$$x = \ldots 1000.10010 \ldots$$
$$\sigma(x) = \ldots 10001.0010 \ldots$$

Subshift: Closed subset $X \subset \mathcal{A}^{\mathbb{Z}}$ that is invariant under the shift map.

Consider the topological dynamical system $\sigma: X \to X$, a homeomorphism of the Cantor set. Can also consider σ-invariant probability measures on X.

SUBSHIFTS

Generally, we can think about the interplay between the language of X the *words* or finite strings of symbols appearing in X and the properties of the dynamical system $\sigma: X \to X$.

DYNAMICAL PROPERTIES OF SUBSHIFTS

Periodic Points: A single word infinitely repeated $x = \ldots www.wwww \ldots$

Dense orbits: Every word that appears in X appears in x.

Transitive: X contains a dense orbit $X = \overline{\mathcal{O}(x)}$.

Minimal: Every orbit in X is dense. (Sturmian system is minimal.)

Recurrent: $X = \overline{\mathcal{O}(x)}$ and if a word w appears in x then w appears infinitely often in x.

COMPLEXITY OF SUBSHIFTS

Consider the sequence $\{c_n(X)\}$ where

$$c_n(X) = \#\{\text{ words of length } n \text{ in } X \}$$

If $X = \{0,1\}^{\mathbb{Z}}$ then $c_n(X) = 2^n$.

If $X \subset \{0,1\}^{\mathbb{Z}}$ is the Sturmian shift then $c_n(X) = n + 1$.

Entropy: $h(X) = \lim_{n\to\infty} \frac{\log c_n(X)}{n}$.

COMPLEXITY OF SUBSHIFTS

Theorem (Morse-Hedlund): If there exists an m such that $c_m(X) \leq m$ then X only consists of a finite number of periodic points.

Proof: If $c_1(X) = 1$ then done, so assume $c_1(X) \geq 2$.

Then we have

$$2 \leq c_1(X) \leq c_2(X) \leq \cdots \leq c_m(X) \leq m.$$

So $c_i(X) = c_j(X)$ for some $i < j$. This implies $\{c_n(X)\}$ is eventually constant.

COMPLEXITY OF SUBSHIFTS

Theorem (Morse-Hedlund): If there exists an m such that $c_m(X) \leq m$ then X only consists of periodic points.

Compare to Sturmian: For all $n \geq 1,\ c_n(X) = n + 1.$

Sturmian is minimal and uniquely ergodic.

How low can complexity be and admit certain properties:
e.g. non-minimal? more than one ergodic measure?

LOW COMPLEXITY SUBSHIFTS

$$X = \{0^\infty, 1^\infty\}$$

$$0^\infty = \ldots 000.0000 \ldots$$

$$1^\infty = \ldots 111.1111 \ldots$$

$$c_n(X) = 2$$

Not minimal, two ergodic measures. Not transitive.

LOW COMPLEXITY SUBSHIFTS

X is the orbit closure of

$$x = \;\ldots 000.1111\ldots$$

$$c_n(X) = n + 1$$

Transitive but not recurrent.

RESULTS ON LOW COMPLEXITY SUBSHIFTS

Boshernitzan (1984): Suppose (X, σ) is a **minimal** subshift.

If

$$\limsup_{n \to \infty} \frac{c_n(X)}{n} < 3$$

then (X, σ) is uniquely ergodic.

RESULTS ON LOW COMPLEXITY SUBSHIFTS

Boshernitzan (1984): Suppose (X, σ) is a **minimal** subshift.

If

$$\liminf_{n\to\infty} \left(c_n(X) - kn\right) = -\infty$$

then (X, σ) has at most $(k-1)$ distinct ergodic invariant measures.

Remark: limsup and liminf generally will not agree here.

RESULTS ON LOW COMPLEXITY SUBSHIFTS

Cyr-Kra (2019): Suppose (X, σ) is **any** subshift.

If

$$\liminf_{n\to\infty} \frac{c_n(X)}{n} < k$$

then (X, σ) has at most $(k-1)$ distinct ergodic **nonatomic** measures.

Atomic measure $\Leftrightarrow$ supported on a periodic orbit.

RESULTS ON LOW COMPLEXITY SUBSHIFTS

O-Pavlov (2019): Suppose (X, σ) is a subshift.

If X is recurrent, transitive, and not minimal then

$$\limsup_{n \to \infty} \left(c_n(X) - \frac{3}{2} n \right) = \infty$$

RESULTS ON LOW COMPLEXITY SUBSHIFTS

Dykstra-O-Pavlov (2022): Suppose (X, σ) is a subshift.

If X is recurrent, transitive, with m distinct infinite minimal subsystems and p distinct periodic subsystems then

$$\limsup_{n\to\infty} \left(c_n(X) - (2m + p + 1)n\right) = \infty$$

RESULTS ON LOW COMPLEXITY SUBSHIFTS

Dykstra-O-Pavlov (2022): Suppose (X, σ) is a subshift.

If X is recurrent and transitive and

$$\limsup_{n \to \infty} \left(c_n(X) - kn\right) = \infty$$

then (X, σ) has at most $(k-1)$ distinct ergodic measures.

EXAMPLE

Consider the following sequence of substitutions:

$$\tau_k : 0 \mapsto 0$$
$$\tau_k : 1 \mapsto 1\underbrace{0 \cdots 0}_{n_k}1$$

Apply the substitutions as follows:

$$\tau_1 \tau_2 \cdots \tau_k(1)$$

Ferenczi (1996): Linear complexity subshifts are S-adic substitutions.

EXAMPLE $\tau_k: 0 \mapsto 0 \quad \tau_k: 1 \mapsto 10^{n_k}1$

$\tau_3: 1 \mapsto 10^{n_3}1$

$\tau_2: 10^{n_3}1 \mapsto 10^{n_2}10^{n_3}10^{n_2}1$

$\tau_1: 10^{n_2}10^{n_3}10^{n_2}1 \mapsto 10^{n_1}10^{n_2}10^{n_1}10^{n_3}10^{n_1}10^{n_2}10^{n_1}1$

Limit is an infinite sequence $\lim_{k\to\infty} \tau_1\tau_2\cdots\tau_k(1)$

$$x = \ldots 10^{n_1}10^{n_2}10^{n_1}10^{n_3}10^{n_1}10^{n_2}10^{n_1}1\ldots$$

EXAMPLE

$$x = \ldots 10^{n_1}10^{n_2}10^{n_1}10^{n_3}10^{n_1}10^{n_2}10^{n_1}1\ldots$$

Assume $n_1 < n_2 < n_3$. Set X equal to the orbit closure of this point (transitive).

Recurrent, not minimal, orbit closure contains fixed point 0^∞.

Given a nondecreasing $g: \mathbb{N} \to \mathbb{R}$, by choosing $\{n_k\}$ to grow sufficiently fast, we obtain an example with

$$c_n(X) < \frac{3}{2}n + g(n)$$

FURTHER RESULTS: CREUTZ - PAVLOV

Creutz-Pavlov proved that systems with $\limsup \frac{c_n(X)}{n} < \frac{3}{2}$ have to be generated by a very specific sequence of substitutions $\{\tau_k\}$.

E.g.

- $\tau_k: 0 \mapsto 0^{m_k - 1}1, \;\; \tau_k: 1 \mapsto 0^{n_k - 1}1$, where $0 < m_k < n_k \leq 2m_k$
- $\tau_k: 0 \mapsto 0^{m_k}10^{r_k}1, \;\; \tau_k: 1 \mapsto 0^{n_k}10^{r_k}1$ with restrictions on m_k, n_k, r_k

In case 1 if $n_k = m_k + 1$ we recover a Sturmian system with rotation number α with continued fraction determined by $\{m_k\}$.

FURTHER RESULTS: CREUTZ-PAVLOV

If (X, σ) is a minimal subshift with $\limsup \frac{c_n(X)}{n} < \frac{3}{2}$

- (X, σ, μ) cannot be weak-mixing, but there is an example $= \frac{3}{2}$
- (X, σ) cannot be topologically mixing, but there are examples arbitrarily close to $\frac{3}{2}$
- (X, σ, μ) cannot be Toeplitz, but there is a Toeplitz example $= \frac{3}{2}$

REFERENCES

- Morse, Marston; Hedlund, Gustav A., *Symbolic Dynamics*, Amer. J. Math.**60** (1938), no.4, 815–866.
- Boshernitzan, Michael, *A unique ergodicity of minimal symbolic flows with linear block growth*, J. Analyse Math. **44** (1984), 77–96.
- Van Cyr and Bryna Kra, *Counting generic measures for a subshift of linear growth*, J. Eur. Math. Soc. (JEMS) **21** (2019), no. 2, 355-380.
- Nic Ormes and Ronnie Pavlov, *On the complexity function for sequences which are not uniformly recurrent*, Dynamical systems and random processes, Contemp. Math., vol. 736, Amer. Math. Soc., [Providence], RI, [2019] ©2019, pp. 125-137.
- Andrew Dykstra, Nicholas Ormes, and Ronnie Pavlov, *Subsystems of transitive subshifts with linear complexity*, Ergodic Theory Dynam. Systems **42** (2022), no. 6, 1967-1993.
- Darren Creutz and Ronnie Pavlov, *On minimal subshifts of linear word complexity with slope less than 3/2*, *https://arxiv.org/abs/2308.14901*