A guide to generalized Nevanlinna-Pick theorems

Rachael M. Norton

Northwestern University

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Recall...

Let $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$.

Bounded analytic function

For an analytic function $f : \mathbb{D} \to \mathbb{C}$, define the *sup norm of f* by $\|f\|_\infty := \sup\{|f(z)| \mid z \in \mathbb{D}\}$. We say f is *bounded* if $\|f\|_\infty < \infty$.

Positive semidefinite matrix

Let A be an $N \times N$ square matrix with entries in $\mathbb{C}$. Assume $A = A^*$, where A^* is the conjugate transpose of A. We say that A is *positive semidefinite* if any of the following are true:

1. $\langle z, Az \rangle \geq 0$ for all $z \in \mathbb{C}^N$.
2. All the eigenvalues of A are nonnegative.
3. All its leading principal minors are nonnegative.

Classical Nevanlinna-Pick Theorem

Theorem (Pick 1915)

Given N distinct points $z_1, \ldots, z_N \in \mathbb{D}$ and N points $\lambda_1, \ldots, \lambda_N \in \mathbb{C}$, there exists an analytic function $f : \mathbb{D} \to \mathbb{C}$ such that $||f||_\infty \leq 1$ and

$$f(z_i) = \lambda_i, \; i = 1, \ldots, N,$$

if and only if the Pick matrix

$$\left[\frac{1-\overline{\lambda_i}\lambda_j}{1-\overline{z_i}z_j}\right]_{i,j=1}^N$$

is positive semidefinite.

Distilled version

Given

- N distinct points in a unit disc
- N points

there exists an interpolating function f with $\|f\| \leq 1 \iff$ a certain matrix is positive semidefinite.

Distilled version

Given

- N distinct points in a unit disc (initial data)
- N points (target data)

there exists an interpolating function f with $\|f\| \leq 1 \iff$ a certain matrix is positive semidefinite (Pick matrix).

Early Generalizations

- (Nagy-Koranyi 1956) Target data in $M_n(\mathbb{C})$.
- (Sarason 1967) Commutant lifting in $H^\infty(\mathbb{D}) = \{f : \mathbb{D} \to \mathbb{C} \mid f \text{ is analytic and bounded}\}$ implies classical Nevanlinna-Pick theorem and Nagy-Koranyi theorem.
- (Ball-Gohberg 1985) Initial data in $M_m(\mathbb{C})$ and target data in $M_n(\mathbb{C})$, proved via commutant lifting.

More generalizations

- (Ball-Gohberg 1985) Initial data in $M_m(\mathbb{C}) = B(\mathbb{C}^m)$ and target data in $M_n(\mathbb{C}) = B(\mathbb{C}^n)$, proved via commutant lifting.
- (Constantinescu-Johnson 2003) Initial data in $B(H)^n$ and target data in $B(H)$, proved via displacement equation.
- (Muhly-Solel 2004) Initial data in a W^*-correspondence and target data in $B(H)$, proved via commutant lifting.

Goal

Understand the relationship between Constantinescu-Johnson's theorem and Muhly-Solel's theorem

1. Understand Constantinescu-Johnson's setting
2. Brief introduction to W^*-correspondences
3. Generalize Constantinescu-Johnson's theorem to the W^*-correspondence setting
4. Compare C-J's theorem with M-S's theorem

Constantinescu-Johnson's setting

Fix

- a Hilbert space H.

Then

- the initial data are in $\{\eta = \begin{bmatrix} \eta_1 \\ \vdots \\ \eta_n \end{bmatrix} \mid \eta_i \in B(H)\}$

- the target data are in $B(H)$

Constantinescu-Johnson's setting

Fix

- a Hilbert space H
- the bimodule $\mathbb{C}^n$ over $\mathbb{C}$
- the homomorphism $\sigma : \mathbb{C} \to B(H)$ given by $\sigma(a) = aI_H$.

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- the initial data are in $\{\eta = \begin{bmatrix} \eta_1 \\ \vdots \\ \eta_n \end{bmatrix} \mid \eta_i \in B(H)\}$
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  $= \{\eta : H \to \mathbb{C}^n \otimes H \mid \eta \circ aI_H = aI_{\mathbb{C}^n \otimes H} \circ \eta\}$
- the target data are in $B(H)$
  $= \{x \in B(H) \mid x\sigma(a) = \sigma(a)x \quad \forall a \in M\}$

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  $= \{\eta : H \to \mathbb{C}^n \otimes H \mid \eta \circ aI_H = aI_{\mathbb{C}^n \otimes H} \circ \eta\}$ (intertwining space)
- the target data are in $B(H)$
  $= \{x \in B(H) \mid x\sigma(a) = \sigma(a)x \quad \forall a \in M\}$

Constantinescu-Johnson's setting

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- a Hilbert space H
- the bimodule $\mathbb{C}^n$ over $\mathbb{C}$
- the homomorphism $\sigma : \mathbb{C} \to B(H)$ given by $\sigma(a) = aI_H$.

Then

- the initial data are in $\{\eta = \begin{bmatrix} \eta_1 \\ \vdots \\ \eta_n \end{bmatrix} \mid \eta_i \in B(H)\}$ $= \{\eta : H \to \mathbb{C}^n \otimes H \mid \eta \circ aI_H = aI_{\mathbb{C}^n \otimes H} \circ \eta\}$ (intertwining space)
- the target data are in $B(H)$ $= \{x \in B(H) \mid x\sigma(a) = \sigma(a)x \quad \forall a \in M\}$ (commutant of $\sigma(\mathbb{C})$ in $B(H)$)

W^*-algebra

Definition

A W^*-**algebra** M is a C^*-algebra that is a dual space. In particular,

- norm $\| \cdot \|$ on M
- involution $*$ on M
- $\|a^* a\| = \|a^*\| \|a\|$ for all $a \in M$

Examples of W^*-algebras

- $\mathbb{C}$
- $M_n(\mathbb{C})$
- $B(H)$, where H is a Hilbert space

W^*-correspondence

Definition

A W^*-**correspondence** E over a W^*-algebra M is

- right Hilbert C^*-module over M
  - right M-module
  - M-valued inner product on E
  - complete w.r.t. norm induced by inner product
- self-dual ($\implies$ all bounded operators on E are adjointable)
- left action of M on E.

Examples of W^*-correspondences

- $M = E = \mathbb{C}$
  - $a \cdot c \cdot b = acb$
  - $\langle c, d \rangle = \overline{c}d$
- $M = \mathbb{C}$, $E = \mathbb{C}^n$
  - $a \cdot \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \cdot b = \begin{bmatrix} ac_1 b \\ \vdots \\ ac_n b \end{bmatrix}$
  - $\left\langle \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}, \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix} \right\rangle = \sum \overline{c_i} d_i$
- Any Hilbert space H is a W^*-correspondence over $\mathbb{C}$
- Any W^*-algebra is a W^*-correspondence over itself

W^*-correspondence setting

Fix

- a Hilbert space H
- a W^*-correspondence E over a W^*-algebra M
- a faithful, normal homomorphism $\sigma : M \to B(H)$

define

- the intertwining space
  $E^\sigma := \{\eta : H \to E \otimes H \mid \eta\sigma(a) = (\varphi(a) \otimes I_H)\eta \quad \forall a \in M\}$
- the commutant
  $\sigma(M)' = \{x \in B(H) \mid x\sigma(a) = \sigma(a)x \quad \forall a \in M\}$

W^*-correspondence setting

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Let E be a W^*-correspondence over the W^*-algebra M, and let $\sigma : M \to B(H)$ be a faithful, normal homomorphism.

Theorem (N. 2017)

Let $\mathfrak{z}_1, \ldots, \mathfrak{z}_N$ be N distinct elements of E^σ with $\|\mathfrak{z}_i\| < 1$ for all i, and let $\Lambda_1, \ldots, \Lambda_N \in \sigma(M)'$. There exists $X \in H^\infty(E^\sigma)$ with $\|X\| \leq 1$ such that

$$X(\mathfrak{z}_i) = \Lambda_i,\ i = 1, \ldots, N,$$

if and only if the operator matrix

$$\left[C(\mathfrak{z}_i)^*(I_{\mathscr{F}(E)} \otimes (I_H - \Lambda_i^*\Lambda_j))C(\mathfrak{z}_j)\right]_{i,j=1}^{N}$$

is positive semidefinite.

Muhly-Solel's theorem

Theorem (Muhly-Solel 2004)

Let $\mathfrak{z}_1, \ldots, \mathfrak{z}_N$ be N distinct elements of E^σ with $\|\mathfrak{z}_i\| < 1$ for all i, and let $\Lambda_1, \ldots, \Lambda_N \in B(H)$. There exists $Y \in H^\infty(E)$ with $\|Y\| \leq 1$ such that

$$Y(\mathfrak{z}_i^*) = \Lambda_i, \quad i = 1, \ldots, N,$$

if and only if the map from $M_N(\sigma(M)')$ to $M_N(B(H))$ defined by

$$[B_{ij}]_{i,j=1}^N \mapsto$$
$$\left[C(\mathfrak{z}_i)^*(I_{\mathscr{F}(E)} \otimes B_{ij})C(\mathfrak{z}_j) - \Lambda_i C(\mathfrak{z}_i)^*(I_{\mathscr{F}(E)} \otimes B_{ij})C(\mathfrak{z}_j)\Lambda_j^*\right]_{i,j=1}^N$$

is completely positive.

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Comparing the generalizations: An implication

Let $\mathfrak{z}_1, \ldots, \mathfrak{z}_N$ be distinct elements of E^σ with $\|\mathfrak{z}_i\| < 1$ for all i, and let $\Lambda_1, \ldots, \Lambda_N \in \sigma(M)'$. If there exists $Y \in H^\infty(E)$ with $\|Y\| \leq 1$ such that

$$Y(\mathfrak{z}_i^*) = \Lambda_i^*, \quad i = 1, \ldots, N$$

in the sense of (Muhly-Solel 2004), then there exists $X \in H^\infty(E^\sigma)$ with $\|X\| \leq 1$ such that

$$X(\mathfrak{z}_i) = \Lambda_i, \quad i = 1, \ldots, N$$

in the sense of (N. 2017). However, a simple example shows that the converse is not true.

Comparing the generalizations: An equivalence

Define $\mathfrak{Z}(E^\sigma) = \{\eta \in E^\sigma \mid a \cdot \eta = \eta \cdot a \quad \forall a \in M\}$.

Theorem (N.)

Let $\mathfrak{z}_1, \ldots, \mathfrak{z}_N$ be N distinct elements of $\mathfrak{Z}(E^\sigma)$ with $\|\mathfrak{z}_i\| < 1$ for all i, and let $\Lambda_1, \ldots, \Lambda_N \in \mathfrak{Z}(\sigma(M)')$. The following are equivalent:

1. There exists $Y \in H^\infty(\mathfrak{Z}(E))$ with $\|Y\| \leq 1$ such that

$$Y(\mathfrak{z}_i^*) = \Lambda_i^*, \quad i = 1, \ldots, N$$

   in the sense of (Muhly-Solel 2004).

2. There exists $X \in H^\infty(\mathfrak{Z}(E^\sigma))$ with $\|X\| \leq 1$ such that

$$X(\mathfrak{z}_i) = \Lambda_i, \quad i = 1, \ldots, N$$

   in the sense of (N. 2017).

Summary of results

- (N. 2017) and Muhly-Solel's theorem are distinct.
- $\Longrightarrow$ Constantinescu-Johnson and Muhly-Solel's theorems are distinct.
- However, when the W^*-correspondence and W^*-algebra are commutative, the theorems yield the same result.

Future work: more generalizations

- In 2017, Jennifer Good proved a generalization of Muhly-Solel's theorem for a weighted Hardy algebra.
- Goal: Generalize (N. 2017) to the setting of the weighted Hardy algebra.

References

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