MS/PhD Qualifying exam: Numerical Analysis
August 22, 2014

Closed book/closed notes.
All questions are equally weighted.
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Part 1 (MATH:5800/22M:170)

1. *Floating point arithmetic*. The standard formula for computing the roots of a quadratic $ax^2 + bx + c = 0$:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

is known to have problems computing the smaller root numerically if b^2 is *much* larger than $|ac|$. Explain the cause of the cause of this problem; propose a method for avoiding this problem.

2. *Solution of nonlinear equations*. Carry out two steps of the secant method for solving $x - \cos x = 0$ starting with $x_0 = 0$ and $x_1 = 1$. What rate of convergence is expected, and under what conditions is this rate of convergence obtained?

3. *Interpolation and approximation*. Using equally spaced interpolation points is known to result in Runge's phenomenon for the function $f(x) = 1/(1 + x^2)$ interpolated over $[-5, +5]$. What is this phenomenon? Can the use of a different set of interpolation points prevent this phenomenon? If so, explain how?

4. *Numerical integration*. Use Simpson's method with five function evaluations to obtain an estimate of $\int_0^1 e^x/(1 + x)\, dx$. What is the asymptotic order of the error of composite Simpson's method with $2n + 1$ function evaluations? Give an example of a method that has an asymptotically faster rate of convergence than Simpson's method as the number of function evaluations goes to infinity.

Part 2 (MATH:5810/22M:171)

1. *Multistep methods*. Consider the general multistep method

$$\boldsymbol{y}_{n+1} = \sum_{j=0}^{p} a_j \boldsymbol{y}_{n-j} + h \sum_{j=-1}^{p} b_j \boldsymbol{f}(t_{n-j}, \boldsymbol{y}_{n-j}).$$

   In order to prove convergence of a particular order for this method we need two basic conditions: a stability condition, and a consistency condition. Give these conditions. Use them to determine if, and with what order, the leap-frog method converges:

$$\boldsymbol{y}_{n+1} = \boldsymbol{y}_{n-1} + 2\,h\,\boldsymbol{f}(t_n, \boldsymbol{y}_n).$$

2. *Runge–Kutta methods*. Show that Heun's method

$$\begin{aligned} \mathbf{z}_{n+1} &= \mathbf{y}_n + h\,\mathbf{f}(t_n, \mathbf{y}_n), \\ \mathbf{y}_{n+1} &= \mathbf{y}_n + \frac{1}{2} h \left[\mathbf{f}(t_n, \mathbf{y}_n) + \mathbf{f}(t_{n+1}, \mathbf{z}_{n+1}) \right] \end{aligned}$$

   has a local truncation error of $\mathcal{O}(h^3)$. What is its asymptotic global truncation error in the form $\mathcal{O}(h^m)$?

3. *LU factorization and linear systems*. The perturbation theorem for linear systems states that if $A\boldsymbol{x} = \boldsymbol{b}$, $(A+E)\widehat{\boldsymbol{x}} = \boldsymbol{b} + \boldsymbol{d}$, and $\left\|A^{-1}\right\| \|E\| < 1$, then

$$\frac{\|\widehat{\boldsymbol{x}} - \boldsymbol{x}\|}{\|\boldsymbol{x}\|} \leq \frac{\kappa(A)}{1 - \kappa(A)(\|E\| \,/\, \|A\|)} \left[\frac{\|E\|}{\|A\|} + \frac{\|\boldsymbol{d}\|}{\|\boldsymbol{b}\|} \right]$$

   where $\kappa(A) = \left\|A^{-1}\right\| \|A\|$ is the condition number. Using this, how many digits of accuracy are expected in the computed solution $\widehat{\boldsymbol{x}}$ given that the matrix A and right-hand side $\boldsymbol{b}$ are known to 5 digits, but $\kappa(A) \approx 10^3$? The backward error theory forLU factorization by Wilkinson shows that the computed solution $\widehat{\boldsymbol{x}}$ of a system $A\boldsymbol{x} = \boldsymbol{b}$ exactly satisfies $(A+E)\widehat{\boldsymbol{x}} = \boldsymbol{b}$ where $\|E\|_\infty \leq 3\mathbf{u} \left(\|A\|_\infty + \left\|\widehat{L}\right\|_\infty \left\|\widehat{U}\right\|_\infty \right)$ where $\widehat{L}$ and $\widehat{U}$ are the computed L and U factors in the LU factorization. If $\left\|\widehat{L}\right\|_\infty \left\|\widehat{U}\right\|_\infty / \|A\|_\infty$ is modest (say ≈ 10), give an estimate for the relative error $\|\widehat{\boldsymbol{x}} - \boldsymbol{x}\|_\infty / \|\boldsymbol{x}\|_\infty$ in terms of $\kappa(A)$ in the ∞-norm.

4. *Least squares and QR factorization*. What is a QR factorization of a matrix? Describe two different ways of computing the QR factorization of an $m \times n$ matrix. Explain how to use a QR factorization of a matrix to solve a least squares problem $\min_{\mathbf{x}} \|A\mathbf{x} - \mathbf{b}\|_2$.