MS/PhD Qualifying exam: Numerical Analysis
August 20, 2020

Closed book/closed notes.
All questions are equally weighted.
Show all working.
Bring a calculator.
No communication devices.

Part 1 MATH:5800

1. *Floating point arithmetic.* In Matlab, the formulas

$$\cos\theta = \frac{x}{\sqrt{x^2+y^2}}, \qquad \text{and}$$
$$\sin\theta = \frac{y}{\sqrt{x^2+y^2}}$$

give $\cos\theta = \sin\theta = 0$ for $x = y = 10^{+200}$. Why does this occur? Can you re-arrange the formulas to give equivalent formulas in exact arithmetic, but give accurate values for $\cos\theta$ and $\sin\theta$.

2. *Solution of nonlinear equations.* Carry out two steps of the secant method for solving $x\,e^x = 3$ starting with $x_0 = 0$ and $x_1 = 1$. What rate of convergence is expected, and under what conditions is this rate of convergence obtained?

3. *Interpolation and approximation.* Using equally spaced interpolation points is known to result in Runge's phenomenon for the function $f(x) = 1/(1+x^2)$ interpolated over $[-5, +5]$. What is this phenomenon? Can the use of a different set of interpolation points prevent this phenomenon? If so, explain how?

4. *Numerical integration.* Use Simpson's method with five function evaluations to obtain an estimate of $\int_0^1 e^x/(1+x)\,dx$. What is the asymptotic order of the error of composite Simpson's method with $2n+1$ function evaluations? Give an example of a method that has an

asymptotically faster rate of convergence than Simpson's method as the number of function evaluations goes to infinity.

Part 2 MATH:5810

1. *Multistep methods*. Consider the general multistep method

$$\begin{aligned} \boldsymbol{y}_{n+1} &= \sum_{j=0}^{p} a_j \boldsymbol{y}_{n-j} + h \sum_{j=-1}^{p} b_j \boldsymbol{f}(t_{n-j}, \boldsymbol{y}_{n-j}) \\ &= \sum_{j=0}^{p} a_j \boldsymbol{y}_{n-j} + h \sum_{j=-1}^{p} b_j \boldsymbol{y}'_{n-j}. \end{aligned}$$

   In order to prove convergence of a particular order for this method we need two basic conditions: a stability condition, and a consistency condition. [**Hint:** The consistency conditions come from Taylor series expansions of $\boldsymbol{y}_{n-j} = \boldsymbol{y}(t_{n-j})$. Give these conditions. Use them to determine if, and with what order, the trapezoidal method converges:

$$\boldsymbol{y}_{n+1} = \boldsymbol{y}_{n-1} + \frac{h}{2} \left[\boldsymbol{f}(t_n, \boldsymbol{y}_n) + \boldsymbol{f}(t_{n+1}, \boldsymbol{y}_{n+1}) \right].$$

   Correction: ***$\boldsymbol{y}_{n-1}$ on the right should be $\boldsymbol{y}_n$.***

$$\boldsymbol{y}_{n+1} = \boldsymbol{y}_{n} + \frac{h}{2} \left[\boldsymbol{f}(t_n, \boldsymbol{y}_n) + \boldsymbol{f}(t_{n+1}, \boldsymbol{y}_{n+1}) \right].$$

   End of correction.

2. *Runge–Kutta methods*. Show that Heun's method

$$\begin{aligned} \mathbf{z}_{n+1} &= \mathbf{y}_n + h\,\mathbf{f}(t_n, \mathbf{y}_n), \\ \mathbf{y}_{n+1} &= \mathbf{y}_n + \frac{1}{2} h \left[\mathbf{f}(t_n, \mathbf{y}_n) + \mathbf{f}(t_{n+1}, \mathbf{z}_{n+1}) \right] \end{aligned}$$

   has a local truncation error of $\mathcal{O}(h^3)$. What is its asymptotic global truncation error in the form $\mathcal{O}(h^m)$?

3. *LU factorization and linear systems*. The perturbation theorem for linear systems states that if $A\boldsymbol{x} = \boldsymbol{b}$, $(A+E)\widehat{\boldsymbol{x}} = \boldsymbol{b}+\boldsymbol{d}$, and $\left\|A^{-1}\right\| \|E\| < 1$, then

$$\frac{\|\widehat{\boldsymbol{x}} - \boldsymbol{x}\|}{\|\boldsymbol{x}\|} \leq \frac{\kappa(A)}{1 - \kappa(A)(\|E\| \,/\, \|A\|)} \left[\frac{\|E\|}{\|A\|} + \frac{\|\boldsymbol{d}\|}{\|\boldsymbol{b}\|}\right]$$

where $\kappa(A) = \left\|A^{-1}\right\| \|A\|$ is the condition number. Using this, how many digits of accuracy are expected in the computed solution $\widehat{\boldsymbol{x}}$ given that the matrix A and right-hand side $\boldsymbol{b}$ are known to 5 digits, but $\kappa(A) \approx 10^3$? The backward error theory forLU factorization by Wilkinson shows that the computed solution $\widehat{\boldsymbol{x}}$ of a system $A\boldsymbol{x} = \boldsymbol{b}$ exactly satisfies $(A+E)\widehat{\boldsymbol{x}} = \boldsymbol{b}$ where $\|E\|_\infty \leq 3\mathbf{u}\left(\|A\|_\infty + \left\|\widehat{L}\right\|_\infty \left\|\widehat{U}\right\|_\infty\right)$ where $\widehat{L}$ and $\widehat{U}$ are the computed L and U factors in the LU factorization. If $\left\|\widehat{L}\right\|_\infty \left\|\widehat{U}\right\|_\infty / \|A\|_\infty$ is modest (say ≈ 10), give an estimate for the relative error $\|\widehat{\boldsymbol{x}} - \boldsymbol{x}\|_\infty / \|\boldsymbol{x}\|_\infty$ in terms of $\kappa(A)$ in the ∞-norm.

4. *QR algorithm*. Give the QR algorithm (with *or* without shifting). Show that if the original matrix A is symmetric, then every iterate of the QR factorization is symmetric. Explain how shifting is used to improve the rate of convergence of the QR algorithm.