MS/PhD Qualifying exam
Numerical Analysis II (22M:171)
August 22, 2014

Closed book/closed notes.
All questions are equally weighted.
Show all working.

1. *Multistep methods*. Consider the general multistep method

$$y_{n+1} = \sum_{j=0}^{p} a_j y_{n-j} + h \sum_{j=-1}^{p} b_j f(t_{n-j}, y_{n-j}).$$

In order to prove convergence of a particular order for this method we need two basic conditions: a stability condition, and a consistency condition. Give these conditions. Use them to determine if, and with what order, the leap-frog method converges:

$$y_{n+1} = y_{n-1} + 2h f(t_n, y_n).$$

2. *Runge–Kutta methods*. The implicit trapezoidal rule is

$$y_{n+1} = y_n + \frac{1}{2} h \left[f(t_n, y_n) + f(t_{n+1}, y_{n+1}) \right].$$

Describe what is meant by the stability region of a Runge–Kutta method. What is the stability region for the implicit trapezoidal method? Rigorously justify your answer.

3. *LU factorization and linear systems*. The perturbation theorem for linear systems states that if $Ax = b$, $(A+E)\widehat{x} = b + d$, and $\left\|A^{-1}\right\| \|E\| < 1$, then

$$\frac{\|\widehat{x} - x\|}{\|x\|} \leq \frac{\kappa(A)}{1 - \kappa(A)(\|E\| / \|A\|)} \left[\frac{\|E\|}{\|A\|} + \frac{\|d\|}{\|b\|} \right]$$

where $\kappa(A) = \left\|A^{-1}\right\| \|A\|$ is the condition number. Using this, how many digits of accuracy are expected in the computed solution $\widehat{x}$ given that the matrix A and right-hand side b are known to 5 digits, but $\kappa(A) \approx 10^3$?

The backward error theory forLU factorization by Wilkinson shows that the computed solution $\widehat{x}$ of a system $Ax = b$ exactly satisfies $(A+E)\widehat{x} = b$ where $\|E\|_\infty \leq 3\mathbf{u}\left(\|A\|_\infty + \left\|\widehat{L}\right\|_\infty \left\|\widehat{U}\right\|_\infty\right)$ where $\widehat{L}$ and $\widehat{U}$ are the computed L and U factors in the LU factorization. If $\left\|\widehat{L}\right\|_\infty \left\|\widehat{U}\right\|_\infty / \|A\|_\infty$ is modest (say ≈ 10), give an estimate for the relative error $\|\widehat{x} - x\|_\infty / \|x\|_\infty$ in terms of $\kappa(A)$ in the ∞-norm.

4. *Least squares, Cholesky and QR factorization*. The normal equations for solving a least squares problem $\min_x \|Ax - b\|_2$ are $A^T Ax = A^T b$. Describe precisely what the Cholesky and QR factorizations are. Show how to solve the least squares problem using either the QR factorization applied to the original system, or Cholesky factorization applied to the normal equations for the least squares problem. Also show that the R matrix in the QR factorization of A is one of the factors in a Cholesky factorization of $A^T A$.