Ph.D. qual. exam./M.S. comp. exam. on Numerical analysis.  
Friday August 18, 2017.

There are two Parts. In Part I, answer at least 5 out of 7 questions. In Part II, answer at least 5 out of 7 questions. Only the best 5 of each Part will count.

Part I, answer at least 5 out of 7 questions:

1. (a) Give the Taylor polynomial $P_2(x)$ of degree 2 about $a = 0$ of the function $f(x) = 1/(1 + x)^{1/3}$.
   
   (b) What is the relative condition number of the function $f(x) = 1/(1 + x)^{1/3}$ at $x = -10^{-9}$?
   
   (c) Give at least 12 correct significant decimal digits of
   $$g(x) = \frac{9}{x} \left( \frac{1}{(1 + x)^{1/3}} - 1 \right)$$
   
   at $x = -10^{-9}$.
   
   (d) Explain succinctly why the calculator gives an inaccurate numerical result when evaluating $g(x)$ directly for $x = -10^{-9}$.

2. Consider the interpolation polynomial $P_n(x)$ of degree $n$ of the function $f(x) = \cos(x)$ on the interval $[0, \pi/2]$ with $n + 1$ Chebyshev nodes $x_j$ on $[0, \pi/2]$.
   
   (a) Give the expression of the $n + 1$ Chebyshev nodes $x_j$ on $[0, \pi/2]$.
   
   (b) These $n + 1$ Chebyshev nodes $x_j$ minimize
   $$\max_{x \in [0, \pi/2]} |\psi_n(x)|$$
   
   where $\psi_n(x)$ is a polynomial of degree $n + 1$ in $x$ depending on the $n + 1$ parameters $x_0, x_1, \ldots, x_n$.
   
   Give $\psi_n(x)$.
   
   (c) How large should $n$ be chosen to ensure that the error on $[0, \pi/2]$ in polynomial interpolation is less than $10^{-6}$, i.e., such that
   $$\|f - P_n\|_\infty = \max_{x \in [0, \pi/2]} |\cos(x) - P_n(x)| \leq 10^{-6}?$$

3. (a) Define what is a cubic spline. In particular what is a periodic cubic spline?
   
   (b) Is the following function on the interval $[0, 2]$ a cubic spline? If yes is it a periodic cubic spline?
   $$s(x) = \begin{cases} 
8 + 2x & \text{for} \quad x \in [0, \frac{1}{2}], \\
7 + 8x - 12x^2 + 8x^3 & \text{for} \quad x \in [\frac{1}{2}, 2]. 
\end{cases}$$

4. We consider 2 nonsymmetric quadrature formulas of order 3 with $s = 2$:
   
   * the Radau IA quadrature formula with weights $(b_1, b_2) = (1/4, 3/4)$ and nodes $(c_1, c_2) = (0, 2/3)$;
   
   * the Radau IIA quadrature formula with weights $(b_1, b_2) = (3/4, 1/4)$ and nodes $(c_1, c_2) = (1/3, 1)$.

   We now consider a composite quadrature formula defined as follows: on each subinterval $[x_j, x_j + h_j]$ we first apply the Radau IA quadrature formula on the subinterval $[x_j, x_j + h_j/2]$ and then we apply the Radau IIA quadrature formula on the subinterval $[x_j + h_j/2, x_j + h_j]$.
   
   (a) Express this composite quadrature formula as a standard quadrature formula on the subinterval $[x_j, x_j + h_j]$ and give its coefficients.
(b) Is this composite quadrature formula symmetric?
(c) What is the order of this composite quadrature formula?

5. (a) Compute the Fourier coefficients \( a_k, b_k, k = 0, 1, 2, \ldots \) of the periodic function \( f(t) = |t| \) on the interval \([-\pi, \pi]\) repeated periodically with period \(2\pi\).

(b) What is the trigonometric polynomial \( S_n(t) \) for \( n = 7 \) minimizing \( \int_{-\pi}^{\pi} (f(t) - S_n(t))^2 dt \)?

6. (a) Define the Legendre polynomials \( P_k(x) \) and give \( P_0(x), P_1(x), \) and \( P_2(x) \).

(b) Find the polynomial \( q_2(x) \) of degree \( \deg(q_2) \leq 2 \) approximating the function \( f(x) = x^{5/3} \) on the interval \([-1,1]\) which minimizes

\[
\int_{-1}^{1} (f(x) - q_2(x))^2 dx.
\]

7. Define quasi-Newton methods to approximate a solution to a system of nonlinear equations \( F(x) = 0 \) with \( F : \mathbb{R}^n \to \mathbb{R}^n \). In particular define the "good" Broyden update and give an explicit formula for the update.

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**Part II**, answer at least 5 out of 7 questions:

1. We consider the following matrix

\[
A = \begin{bmatrix} 16 & -8 & 4 \\
-8 & 13 & 4 \\
4 & 4 & 9 \end{bmatrix} \in \mathbb{R}^{3 \times 3}.
\]

(a) Calculate by hand the matrix factor \( R \) of the Cholesky decomposition of \( A \).

(b) Give the exact value of the determinants \( \det(R) \) and \( \det(A) \).

(c) Give the exact values of \( |||A|||_1 \) and \( |||A|||_{\infty} \).

(d) Is the matrix \( A \) is (strictly) positive definite (you will be given no credit if you base your proof on the eigenvalues of \( A \) obtained with a calculator)?

2. (a) Define Gauss-Seidel iterations to approximate the solution to a system of linear equations \( Ax = b \) with matrix \( A \in \mathbb{R}^{n \times n} \) and \( b \in \mathbb{R}^n \)

(b) Consider the system of linear equations

\[
\begin{bmatrix} 4 & 1 \\
8 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\
x_2 \end{bmatrix} = \begin{bmatrix} 2 \\
3 \end{bmatrix}.
\]

Starting from the initial vector

\[
x^{(0)} = \begin{bmatrix} x_1^{(0)} \\
x_2^{(0)} \end{bmatrix} := \begin{bmatrix} 1 \\
2 \end{bmatrix},
\]

do Gauss-Seidel iterations converge to the solution \( x^* \) of the system of linear equations (1), i.e., do we have \( \lim_{k \to \infty} x^{(k)} = x^* \)? Prove convergence or divergence (computing the first few iterates is not a proof).

3. Consider the linear least squares problem

\[
\min_{x \in \mathbb{R}^n} ||b - Ax||_2
\]

where \( A \in \mathbb{R}^{m \times n} \) with \( m \geq n \), \( b \in \mathbb{R}^m \), and \( x \in \mathbb{R}^n \).

(a) Give the normal equations that a minimizer \( x^* \) of (2) must satisfy.
(b) Suppose $A \in \mathbb{R}^{m \times n}$ is of rank $n$ and that we have a QR decomposition of $A$ with $Q \in \mathbb{R}^{m \times m}$ orthogonal and $R \in \mathbb{R}^{m \times n}$ upper triangular, show how to obtain a minimizer $x^*$ of (2) using this QR decomposition of $A$.

(c) Find $x \in \mathbb{R}^2$ minimizing $\|b - Ax\|_2$ for

$$A := \begin{bmatrix} 1 & -2 \\ 2 & 1 \\ 1 & 1 \end{bmatrix} \in \mathbb{R}^{3 \times 2}, \quad b := \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix} \in \mathbb{R}^3.$$  

4. Consider the system of ODEs $\dot{x} = f(t, x)$ and the following explicit Runge-Kutta method

\[ \begin{align*}
X_1 &= x_0, \\
X_2 &= x_0 + h \frac{1}{2} f(t_0, X_1), \\
X_3 &= x_0 + h \left( -f(t_0, X_1) + 2f(t_0 + \frac{h}{2}, X_2) \right), \\
x_1 &= x_0 + h \left( \frac{1}{8} f(t_0, X_1) + \frac{3}{8} f(t_0 + \frac{h}{2}, X_2) + \frac{1}{2} f(t_0 + h, X_3) \right).
\end{align*} \]

(a) What is the local order $p$ of this method?

(b) What is the (linear) stability function $R(z)$ of this method ($\dot{z} = \lambda z$ and $z := h\lambda$)?

5. We consider the following implicit linear multistep method applied to $\dot{x} = f(t, x)$ with stepsize $h$ (using the notation $f_j := f(t_j, x_j)$)

$$x_{n+1} = x_n + \frac{h}{12} (5f_{n+1} + 8f_n - f_{n-1}).$$

(a) What is the local order of this method?

(b) Is it 0-stable?

(c) Is it globally convergent?

6. What is a Householder matrix $H \in \mathbb{R}^{n \times n}$? What is a matrix $M \in \mathbb{R}^{n \times n}$ in Hessenberg form? Explain in detail the first step of the reduction of a matrix $A \in \mathbb{R}^{n \times n}$ to Hessenberg form by using a Householder matrix.

7. To find the eigenvalues of a matrix $A \in \mathbb{C}^{n \times n}$ write down the QR algorithm with single shift. Show that if $A \in \mathbb{C}^{n \times n}$ is Hermitian ($A^* = A$) then all $T_k$ for $k = 0, 1, 2, 3, \ldots$ (of the QR algorithm with single shift) are also Hermitian ($T_k^* = T_k$).