Qualifying Exam for Math 5600
August 19, 2020
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INSTRUCTION:

• The questions for this exam (Math 5600) are divided into two parts.

Answer both questions in Part I.

Answer only one question in Part II.

• If you work on more than one question in Part II, please state clearly which one should be graded.

• No additional credit will be given for more than one of the questions in Part II.

• If no choice between the questions is indicated, then the first optional question attempted will be the only one graded.

• All the questions have equal points.

• Please start a new page for every new question and put your name on each sheet.

• Please turn in the exam questions with your solutions.

• Please turn in the scratch papers. All scratch papers will be discarded.

Good Luck!
Part I. Please answer **BOTH** questions 1 and 2.

**Question 1.** Consider the motion of an undamped harmonic oscillator, given by the equation

\[ \ddot{x} = -4x, \]

where \( x(t) \) represents the location of the oscillator, and \( \ddot{x} \) denotes the second derivative of \( x \) with respect to \( t \).

(a) Introduce a new variable \( y \) for the velocity of the oscillator and formulate the motion of the harmonic oscillator as a two dimensional linear system of the form \( \dot{X} = AX \), where \( X = (x, y)\top \) and \( A \) is a \( 2 \times 2 \) matrix.

(b) Find the eigenvalues and eigenvectors of \( A \).

(c) Show there exists an invertible matrix \( T \) so that \( T^{-1}AT = B \) where \( B \) is in Jordan canonical form.

(d) Use the Jordan canonical form to find the general solution of the system \( \dot{X} = AX \).

**Question 2.** Consider the system

\[ \dot{x} = y - ax \]
\[ \dot{y} = -ay + \frac{x}{1 + x}, \]

where \( a \) is a positive parameter. Answer the following questions.

(a) For each qualitatively different value of \( a > 0 \), find all equilibrium points. When the Hartman-Grobman theorem applies, classify each equilibrium point.

(b) Describe the bifurcation that occurs as \( a \) varies and find the critical value of \( a \) (call it \( a^* \)) at which the bifurcation occurs. (You do not need to compute the center manifold at \( a = a^* \).)

(c) Sketch the phase plane (phase portrait) for \( a > a^* \) which qualitatively describes the full dynamics of the system. *Hint: You should indicate the equilibrium points, a heteroclinic orbit, the stable and unstable curves, and six trajectories.*
Part II. Please answer ONLY ONE of the following questions.

Question 3. Consider
\[ \begin{align*}
\dot{x} &= y - x \\
\dot{y} &= x - y - xz \\
\dot{z} &= xy - z.
\end{align*} \]

(a) State the definitions of a (Lyapunov) stable equilibrium point and an asymptotically stable equilibrium point.

(b) Is the origin \((0,0,0)^T\) a (Lyapunov) stable equilibrium point or asymptotically stable equilibrium point?

(c) What is the basin of attraction?

Hint: Find an appropriate Lyapunov function and use LaSalle’s invariance principle.

Question 4. Consider the system
\[ \begin{align*}
\dot{x} &= x(1 - r) - y \\
\dot{y} &= y(1 - r) + x,
\end{align*} \]
where \(r^2 = x^2 + y^2\). This system has a periodic solution \(\gamma(t) = (\cos(t), \sin(t))^T\). Determine the stability of \(\gamma(t)\).

Hint: You may use ONE (and only one) of the following methods:

1. Compute the characteristic multipliers for \(\gamma(t)\). (Liouville’s Formula may help.)
2. Compute the Poincaré map of \(\gamma(t)\). (Polar coordinate transformation may help.)