Qualifying Exam for Math 5600

August 19, 2020

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INSTRUCTION:

- The questions for this exam (Math 5600) are divided into two parts.

Answer both questions in Part I.

Answer only one question in Part II.

- If you work on more than one question in Part II, please state **clearly** which one should be graded.
- No additional credit will be given for more than one of the questions in Part II.
- If no choice between the questions is indicated, then the first optional question attempted will be the only one graded.
- All the questions have equal points.
- Please start a new page for every new question and put your name on each sheet.
- Please turn in the exam questions with your solutions.
- Please turn in the scratch papers. All scratch papers will be discarded.

Good Luck!

Part I. Please answer BOTH questions 1 and 2.

Question 1. Consider the motion of an undamped harmonic oscillator, given by the equation

$$\ddot{x} = -4x,$$

where $x(t)$ represents the location of the oscillator, and $\ddot{x}$ denotes the second derivative of x with respect to t.

(a) Introduce a new variable y for the velocity of the oscillator and formulate the motion of the harmonic oscillator as a two dimensional linear system of the form $\dot{X} = AX$, where $X = (x, y)^\top$ and A is a 2×2 matrix.

(b) Find the eigenvalues and eigenvectors of A.

(c) Show there exists an invertible matrix T so that $T^{-1}AT = B$ where B is in Jordan canonical form.

(d) Use the Jordan canonical form to find the general solution of the system $\dot{X} = AX$.

Question 2. Consider the system

$$\dot{x} = y - ax$$
$$\dot{y} = -ay + \frac{x}{1+x},$$

where a is a positive parameter. Answer the following questions.

(a) For each qualitatively different value of $a > 0$, find all equilibrium points. When the Hartman-Grobman theorem applies, classify each equilibrium point.

(b) Describe the bifurcation that occurs as a varies and find the critical value of a (call it a^*) at which the bifurcation occurs. (You do not need to compute the center manifold at $a = a^*$.)

(c) Sketch the phase plane (phase portrait) for $a > a^*$ which qualitatively describes the full dynamics of the system. *Hint: You should indicate the equilibrium points, a heteroclitic orbit, the stable and unstable curves, and six trajectories.*

Part II. Please answer ONLY ONE of the following questions.

Question 3. Consider

$$\begin{aligned}\dot{x} &= y - x\\ \dot{y} &= x - y - xz\\ \dot{z} &= xy - z.\end{aligned}$$

(a) State the definitions of a (Lyapunov) stable equilibrium point and an asymptotically stable equilibrium point.

(b) Is the origin $(0,0,0)^\top$ a (Lyapunov) stable equilibrium point or asymptotically stable equilibrium point?

(c) What is the basin of attraction?

Hint: Find an appropriate Lyapunov function and use LaSalle's invariance principle.

Question 4. Consider the system

$$\begin{aligned}\dot{x} &= x(1-r) - y\\ \dot{y} &= y(1-r) + x,\end{aligned}$$

where $r^2 = x^2 + y^2$. This system has a periodic solution $\gamma(t) = (\cos(t), \sin(t))^\top$. Determine the stability of $\gamma(t)$.

Hint: You may use ONE (and only one) of the following methods:

1. *Compute the characteristic multipliers for $\gamma(t)$. (Liouville's Formula may help.)*
2. *Compute the Poincaré map of $\gamma(t)$. (Polar coordinate transformation may help.)*