Abstract

In this talk I want to present an application of the so-called 'Young Measures' to the study of the existence of weak solutions to certain PDEs, more precisely, non-linear hyperbolic scalar conservation laws.

A nonlinear scalar conservation law is a PDE of the form

\[ u_t + f(u)_x = 0 \quad (x, t) \in \mathbb{R} \times \mathbb{R}^+, \]

where \( f : \mathbb{R} \to \mathbb{R} \) is a smooth function that we assume to be convex.

Equations sharing this structure arise in diverse contexts in the study of fluid dynamics, traffic flow models and bio mathematics. One of the most remarkable features of these problems is the formation of singularities in finite time ([Lax, 1972]) and these sort of difficulties require us to implement non standard techniques to study these equations.

The implementation of Young measures to study these problems was introduced for the first time by [Tartar, 1979] and [Murat, 1978] in the framework of the theory of Compensated Compactness and the method of the vanishing viscosity. Roughly speaking, the method consists in performing parabolic perturbations to the equation (1) of the form

\[ u_t + f(u)_x = \epsilon u_{xx} \quad (x, t) \in \mathbb{R} \times \mathbb{R}^+, \]

where \( \epsilon \) is a parameter that we call 'viscosity coefficient'. The implementation of this second order viscous term in (2) transform our original equation to a parabolic equation that can be studied using classical tools from the theory of parabolic equations ([Smoller and Hoff, 1985]). In this context, is now possible to find a sequence of global smooth solutions \( \{u^\epsilon\} \) as the coefficient \( \epsilon \) varies. Under certain conditions over the flux \( f \), it is possible to show that the sequence \( \{u^\epsilon\} \) converges to a weak solution of the problem (1) and the Young measures play an important role in this process as the the following theorem illustrates it.

**Theorem 1.** Let \( \Omega \) be an open set of \( \mathbb{R}^n \) y assume that \( K \subset \mathbb{R}^m \) is bounded. Let \( \{u^\epsilon\} \) be a sequence of measurable functions

\[ u^\epsilon : \Omega \to \mathbb{R}^m \]

such that \( u^\epsilon(x) \in K \) for almost all \( x \in \Omega \). Then there exists a subsequence \( \{u^{\epsilon_k}\} \) and a family of probability measures on \( \mathbb{R}^m \), \( \{\nu_x\}_{x \in \Omega} \) with \( \text{supp} \nu_x \subset \overline{K} \) such that if \( f \in C(\mathbb{R}^m) \) and

\[ f(x) = \langle \nu_x, f(\lambda) \rangle = \int_{\mathbb{R}^m} f(\lambda) \, d\nu_x(\lambda), \]

then

\[ f(u^{\epsilon_k}) \rightharpoonup f. \]

The measures obtained in the previous result are called **Young parametrized measures** and with additional conditions over the sequence \( \{u^\epsilon\} \) in \( W^{-1,2}(\Omega) \) it is possible to show that there exists a weak limit \( u \) that turns out to be a solution for the initial problem (1)([Frid, 1993]).

**Theorem 2.** Let \( u_0 \in W^{2,2}(\mathbb{R}) \) and \( f \in C^{2}(\mathbb{R}) \). Then the problem

\[
\begin{align*}
  u_t + f(u)_x &= 0, & (x, t) &\in \mathbb{R} \times \mathbb{R}^+, \\
  u(x, 0) &= u_0(x), & x &\in \mathbb{R}.
\end{align*}
\]

has a weak solution \( u \in L^\infty(\mathbb{R} \times \mathbb{R}^+) \).
References


