1. Find the solution to the initial value problem

\[ u_t + \left( \frac{u^2}{2} \right)_x = 0, \quad x \in \mathbb{R}, \quad t > 0, \]

\[ u(x, 0) = \begin{cases} 
2 & x \leq 0 \\
2 - x & 0 < x \leq 2 \\
1 & 2 < x 
\end{cases} . \]
2. Show that if the $C^1$ initial data $f(x)$ has $f'(x_0) < 0$ for some $x_0$ and that $F''(u) \geq 1$ for all $u$, then the $C^1$ solution of

$$u_t + F(u)_x = 0, \quad u(x, 0) = f(x)$$

must break down at some time $t > 0$. 
3. Let $u(x,t)$ and $v(x,t)$ be solutions of the equation

$$u_t - ku_{xx} = q(x,t), \quad x \in R, \quad 0 < t \leq T$$

satisfy

$$u(x,0) = f(x), \quad v(x,0) = g(x), \quad x \in R$$

respectively, where $k > 0, T > 0$, and $f(x), g(x), q(x,t)$ are continuous and bounded functions. Suppose that $u(x,t)$ and $v(x,t)$ are continuous and bounded on $x \in R, \ 0 \leq t \leq T$, and that

$$f(x) \leq g(x), \quad x \in R.$$ 

Show that $u(x,t) \leq v(x,t)$ for $x \in R, \ 0 \leq t \leq T$. 

4. Solve the initial-boundary-value problem
\[ u_t = u_{xx}, \quad 0 < x < 1, \quad t > 0, \]
\[ u(0, t) = 0, \quad u(1, t) = 1, \quad t > 0, \]
\[ u(x, 0) = x^2, \quad 0 < x < 1. \]
Also find a steady-state solution \( U(x) \) of the above problem.
5. Consider the damped wave equation problem

\[ u_{tt} + du_t - c^2 u_{xx} = 0, \quad x \in \mathbb{R}, \quad t > 0, \]

\[ u(x, 0) = f(x), \quad u_t(x, 0) = g(x), \quad x \in \mathbb{R} \]

where \( c > 0, \ d > 0 \) and \( f, \ g \) are smooth functions with compact support.

Define energy as

\[ e(t) = \frac{1}{2} \int_{\mathbb{R}} (u_t^2(x, t) + c^2 u_x^2(x, t)) \, dx. \]

Show that the energy is nonincreasing as \( t \) increases.