Qualifying Exam: PDE, Spring, 2019

Choose any **Four** out of the five problems. Please indicate your choice. Show all your work.

1. Show that if the $C^1$ initial data $f(x)$ has $f'(x_0) < 0$ for some $x_0$, then the $C^1$ solution of

$$u_t + (u^2)_x = 0, \quad x \in \mathbb{R}, t > 0, \quad u(x, 0) = f(x)$$

must break down at some time $t > 0$.  

2. (i) Solve the initial value problem
\[ u_t + x^2 u_x = -u, \quad x \in \mathbb{R}, \ t > 0, \]
\[ u(x, 0) = x^2, \quad x \in \mathbb{R}. \]
(ii) Over which region in the x-t plane does the solution exist?
(iii) Write an upwind scheme for the above problem.
3. Solve the following initial boundary value problem
\[ u_t - u_{xx} = 0, \quad x > 0, \quad t > 0, \]
\[ u(x, 0) = f(x), \quad x \geq 0, \]
\[ u(0, t) = 1, \quad t \geq 0 \]
where \( f \in C^2[0, +\infty) \) is bounded and \( f(0) = 1 \).
4. (i) Solve the initial-boundary-value problem
\[ u_t = u_{xx}, \quad 0 < x < 1, \quad t > 0, \]
\[ u(0, t) = 0, \quad u(1, t) = 3, \quad t > 0, \]
\[ u(x, 0) = x^2 + 2x, \quad 0 \leq x \leq 1. \]
(ii) What is the limit of the solution as \( t \to +\infty? \)
5. Consider the damped wave equation problem
\[ u_{tt} + du_t - c^2 u_{xx} = 0, \quad x \in \mathbb{R}, \ t > 0, \]
\[ u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad x \in \mathbb{R} \]
where \( c > 0, \ d > 0 \) and \( f, g \) are smooth functions with compact support.

Define energy as 
\[ e(t) = \frac{1}{2} \int_{\mathbb{R}} (u_t^2(x,t) + c^2 u_x^2(x,t)) \, dx. \]

Show that the energy is nonincreasing as \( t \) increases.