Ph.D. Qualifying Exam in Topology
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Instructions. Do eight problems, four from each part. **That is four from part A and four from part B.** This is a closed book examination, you should have no books or paper of your own. Please do your work on the paper provided. Clearly number your pages corresponding to the problem you are working. When you start a new problem, start a new page; only write on one side of the paper. Make a cover page and indicate clearly which eight problems you want graded.

Always justify your answers unless explicitly instructed otherwise. You may use theorems if the problem is not a step in proving that theorem, but you need to state any theorems you use carefully.

Part A

Assume $\mathbb{R}^n$ has the usual topology unless stated otherwise.

1. Let Σ be an oriented closed surface of genus g. Consider the group action ρ of S_g on the g-fold product space $\Sigma \times \cdots \times \Sigma$ as

$$\rho(\sigma) : (x_1, \cdots, x_g) \mapsto (x_{\sigma(1)}, \cdots, x_{\sigma(g)}).$$

   Consider the quotient space $\Sigma \times \cdots \times \Sigma / S_g$ which is denoted by $\mathrm{Sym}^g(\Sigma)$. Let $\alpha_1, \cdots, \alpha_g, \beta_1, \cdots, \beta_g$ be embedded, pairwise distinct circles in Σ. Consider subspaces of $\mathrm{Sym}^g(\Sigma)$

$$\mathbb{T}_\alpha = \alpha_1 \times \cdots \times \alpha_g \text{ and } \mathbb{T}_\beta = \beta_1 \times \cdots \times \beta_g.$$

   (a) What is the dimension of $\mathrm{Sym}^g(\Sigma)$?

   (b) Describe the intersection $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$.

2. Let $X \approx D^2 \times S^1$ be an unknotted solid torus embedded in S^3. Let $T := \partial X \approx S^1 \times S^1$ and a meridian $\mu := S^1 \times \{\cdot\}$ and a longitude $\lambda := \{\cdot\} \times S^1$. Let $\phi : T \to T$ be a diffeomorphism whose induced homomorphism $\phi_* : \pi_1(T) \to \pi_1(T)$ of the fundamental group takes the homotopy class $[\mu]$ to the homotopy class $p[\mu] + q[\lambda]$. The Lens space $L(p, q)$ is a closed manifold obtained from S^3 removing T from S^3 and then gluing back T using the diffeomorphism ϕ. Find the fundamental group of the lens space $L(p, q)$.

3. Let $X \approx D^2 \setminus \{p, q, r\}$ be a trice-punctured disk. Find a 2-fold cover $\tilde{X}$ of X, and describe how the covering transformation group acts on $\tilde{X}$.

4. Let X be a non-empty compact Hausdorff space. If X has no isolated points, show that X is uncountable.

5. Let X be a Δ-complex.

   - Define the homology groups $H_i^\Delta(X)$, $i \geq 0$, of X.
   - Let x_0 be a vertex of X. Construct a map

$$Hurewicz : \pi_1(X, x_0) \to H_1^\Delta(X).$$

   - Prove that *Hurewicz* is well-defined.

6.
- Define the product topology $\mathcal{P}$ of a product of topological spaces.
- Define the box topology $\mathcal{B}$ of a product of topological spaces.
- Let $\mathbb{R}^\infty$ denote a countable product of copies of the real line. Prove that the box and product topologies are distinct: $(\mathbb{R}^\infty, \mathcal{P})$ is *not* homeomorphic to $(\mathbb{R}^\infty, \mathcal{B})$.

Part B

1. Let F be a compact 2-dimensional manifold with connected boundary. Let $h : F \to F$ be an orientation preserving diffeomorphism fixing the boundary ∂F pointwise.

   - Give an example of map h that is not isotopic to the identity.

   Let W be the 3-manifold obtained by taking the product $F \times [0,1]$ and identifying point $(x,1)$ with $(h(x),0)$ for every point $x \in F$. Let θ be a coordinate of $\partial F = S^1$. Let

   $$\pi : W \to S^1; (x,\phi) \mapsto \phi$$

   be a fiber bundle.

   Let $A \subset F$ be a collar neighborhood of ∂F. Let (θ,t) be coordinates of A such that $\{(\theta,t)|t=0\} = \partial F$. Let $\alpha \in \Omega^1(F)$ a differential 1-form on F such that $d\alpha$ is a volume form on F and $\alpha = (1+t)d\theta$ on A.

   - Evaluate the integral of $d\alpha \in \Omega^2(F)$ over the entire manifold F.
   - Let $\omega := \alpha + \kappa\pi^*(d\phi)$ where $\kappa \in \mathbb{R}$ is a constant. For large $\kappa \gg 1$, show that $\omega \wedge d\omega > 0$.

2. - Find a 2-form ω on $\mathbb{R}^{2n}$ such that $\omega^n = \omega \wedge \cdots \wedge \omega > 0$.
   - Let (r,θ,z) be the cylindrical coordinates of $\mathbb{R}^3$. Let $f_i(r)$ be a smooth functions for $i = 1,2$. Let $w = f_1(r)d\theta + f_2(r)d\phi$. Find a necessary and sufficient condition on f_1 and f_2 such that $w \wedge dw = 0$.

3. Find a smooth function on $\mathbb{R}^5$ such that $(0,1,2,3,4) \in \mathbb{R}^5$ serves as a critical point of index 3.

4. Let X be a 4-dimensional smooth manifold and ω is a closed, non-degenerate differential 2-form on X. Let $H : X \to \mathbb{R}$ be a smooth function and $a \in \mathbb{R}$ be a regular value of H. Let $Y := H^{-1}(a)$ and

$$L_Y := \{v \in TX \mid \omega(v, x) = 0 \text{ for all } x \in TY\}$$

Let v_H be a vector field on X specified by $dH = \iota_{v_H}\omega$.

- Show that $L_Y \subset TY$.
- Show that $v_H|_Y \in L_Y$.
- Show that v_H is uniquely determined.

5. Prove that $SL(n, \mathbb{R}) = \{A \in Mat_{n\times n}(\mathbb{R}) : \det(A) = 1\}$ is a manifold and find its dimension.

6. - Define section of vector bundle
   - Give an example of vector bundle that does not admit nowhere vanishing sections, and prove it.
   - Prove that differential k-forms $\Omega^k(M)$ on a smooth n-manifold M are sections of a vector bundle $E \to M$. What is E?