Ph.D. Qualifying Exam in Topology

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Instructions. Do eight problems, four from each part. **That is four from part A and four from part B.** This is a closed book examination, you should have no books or paper of your own. Please do your work on the paper provided. Clearly number your pages corresponding to the problem you are working. When you start a new problem, start a new page; only write on one side of the paper. Make a cover page and indicate clearly which eight problems you want graded.

Always justify your answers unless explicitly instructed otherwise. You may use theorems if the problem is not a step in proving that theorem, but you need to state any theorems you use carefully.

Part A

Assume $\mathbb{R}^n$ has the usual topology unless otherwise stated.

1. Suppose that M and N are smooth n-manifolds and $f : D^n \to M$, $g : D^n \to N$ are smooth embeddings of the disk $D^n = \{x \in \mathbb{R}^n : x \cdot x \leq 1\}$. Form a new space

$$M \# N := \frac{M^\circ \cup N^\circ}{f(x) \sim g(x) \text{ when } x \in \partial D^n}$$

where $M^\circ := M \backslash f(int(D^n))$ and $N^\circ = N \backslash g(int(D^n))$.

   - Compute $\pi_1(M \# N)$ in terms of $\pi_1(M)$ and $\pi_1(N)$.
   - As stated, is this operation independent of the choices of f and g? Prove or disprove.

2. Let $X := \mathbb{R}^2 \backslash \mathbb{Q}^2$. Prove that the fundamental group $\pi_1(X)$ is uncountable.

3. Prove that any open cover of a compact metric space has a Lebesgue number.

4. Prove that the definition of a continuous map $f : X \to Y$ between two topological spaces X and Y is equivalent to the definition of a continuous map $f : (X, d_X) \to (Y, d_Y)$ between metric spaces (X, d_X) and (Y, d_Y) when the topology on X and Y is the metric topology.

5. Let Δ^2 be the 2-simplex and $\partial\Delta^2$ be the boundary of Δ^2. Compute the simplicial homology of the N-fold wedge product $H_*(\bigvee_{i=1}^{N} \partial\Delta^2)$ of the boundaries.

6. Let $W = S^1 \wedge S^1$ be the wedge of two circles. Construct a covering space $\pi : \tilde{W} \to W$ with deck transformation group $\mathbb{Z} \times \mathbb{Z}$.

Part B

1. Describe when a closed manifold M is orientable. Prove that for any closed manifold M, its tangent space TM is orientable.

2. Let $M = \{(x, y, 0) \in \mathbb{R}^3 \mid (x, y) \neq (0, 0)\}$. Let $\omega = \frac{1}{x^2+y^2}(xdy - ydx)$ be a 1-form on M.

   (a) Is M a smooth manifold?

   (b) Is ω exact on M?

   (c) Is ω closed on M?

3. Let $S^2 \subset \mathbb{R}^3$ be the sphere centered at the origin with radius r. Find the (maximal) subset of S^2 where $(dx \wedge dy)/z = (dy \wedge dz)/x$ holds.

4. (a) State the Stokes' theorem in general form (i.e., n-dimensional).

   (b) Deduce the Stokes' theorem in vector calculus from (a).

   (c) Deduce the Green's theorem in vector calculus from (a).

   (d) Deduce the Fundamental theorem of Calculus from (a).

5. Let α, β be two smooth 1-forms on S^3, prove that

$$\int_{S^3} \alpha \wedge d\beta = \int_{S^3} d\alpha \wedge \beta.$$

6. Let $M_{n\times n}$ be the space of all real valued $n \times n$ matrices. Describe its tangent space at identity matrix. Consider the map

$$\det : \mathrm{M}_{\mathrm{n\times n}} \to \mathbb{R}$$

   by sending a matrix A to its determinant $\det(A)$. Compute the differential of this map when A is the identity matrix.