Instructions. Be sure to put your name on each booklet you use.

Much of this examination is “true-false”. When a problem begins with “True-false”, you are to decide if the operative assertion is true or false. If you decide that it is “true”, you are to give a proof, while if you decide that it is “false”, you are to present a counter example.

The exam is divided into two parts. The first covers real analysis and the second covers complex analysis. Each part has 5 problems. You need only work 4 problems in each part. You must indicate which 4 you are submitting for evaluation. If you want to do five in a part, that is OK. We will treat the extra problem as a bonus.

Part I
1. True-false. Lebesgue measure is continuous in the sense that the Lebesgue measure of the closure of a set coincides with the Lebesgue measure of the set.

2. True-false. Let $I_1$ and $I_2$ be two disjoint open intervals, and for $i = 1, 2$, let $A_i$ be an arbitrary subset of $I_i$. Then $m^*(A_1 \cup A_2) = m^*(A_1) + m^*(A_2)$, where $m^*$ denotes Lebesgue outer measure.

3. True-false. Let $\{f_n\}_{n \geq 0}$ be a sequence of non-negative integrable functions defined on $\mathbb{R}$ such that

$$0 \leq f_1(x) \leq f_2(x) \leq f_3(x) \leq \ldots$$

and such that the sequence of numbers $\{\int_{\mathbb{R}} f_n(x) \, dx\}$ is bounded. Let $f(x) = \lim_{n \to \infty} f_n(x)$. Then $f(x) < \infty$ for almost all $x$.

4. True-false. Let $\sigma$ be the function on $\mathbb{R}$ that is zero to the left of 1, 1/2 for $1 \leq x < 2$, 3/4 for $2 \leq x < 3$, 7/8 for $3 \leq x < 4$, etc. (Thus, $\sigma$ jumps by $2^{-n}$ at $n$ for $n = 1, 2, \ldots$, is constant between any two consecutive integers and is continuous from the right.) If $f(x) = x$ on $\mathbb{R}$, then $f$ is integrable with respect to the Lebesgue-Stieltjes measure determined by $\sigma$.

5. Let $\{f_n\}$ be a uniformly bounded sequence of measurable functions defined on the interval $[0, 1]$ and let

$$F_n(x) = \int_0^x f_n(t) \, dt \quad 0 \leq x \leq 1.$$ 

Show that there is a subsequence $\{F_{n_k}\}$ that converges uniformly on $[0, 1]$.

Part II
1. True-false. Suppose $f$ is analytic in the region $0 < |z| < 1$ and suppose that for each $r$, $0 < r < 1$, the integral $\int_{C_r} f(z) \, dz = 0$, where $C_r$ is the circle $|z| = r$. Then $f$ is analytic on the open unit disc.

2. Suppose $f$ is analytic in the annular region $1 - \epsilon < |z| < 2 + \epsilon$ for some positive $\epsilon$. Suppose also that $|f| \leq 1$ on the circle $|z| = 1$ and that $|f| \leq 4$ on the circle $|z| = 2$. Show that $|f(z)| \leq |z|^2$ for all $z$, $1 < |z| < 2$.

3. True-false. Let $\mathcal{G}$ be a domain and let $z_0$ be a point in $\mathcal{G}$. Suppose $f$ is analytic in $\mathcal{G}/\{z_0\}$ and that $f$ takes values in the upper half-plane. Then $z_0$ is a removable singularity of $f$.

4. Find the Laurent series representation of the function

$$f(z) = \frac{1}{z^2(1-z)}$$

that is valid in the region $1 < |z| < \infty$. 

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5. Let \( f \) be analytic in the open unit disc \( \mathbb{D} \) and let the Taylor series expansion for \( f \) be

\[
f(z) = \sum_{n=0}^{\infty} a_n z^n
\]

Suppose

(a) \( f(\mathbb{D}) \subseteq \mathbb{D} \)
(b) \( a_0 = 0 \)
(c) \( |a_1| = 1 \)

Calculate \( \sup\{|a_n| \mid n \geq 2\} \).