Ph.D. Qualifying Examination in Analysis
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Instructions. Be sure to put your name on each booklet you use.

Much of this examination is "true-false". When a problem begins with "True-false", you are to decide if the operative assertion is true or false. If you decide that it is "true", you are to give a proof, while if you decide that it is "false", you are to present a counter example or explain why it is false.

The exam is divided into two parts. The first tests your knowledge of real analysis and the second tests your knowledge of complex analysis. Each part has 5 problems. You need only work 4 problems in each part. Please indicate which 4 you are submitting for evaluation. If you want to do five in a part, that is OK. We will treat the extra problem as a bonus to be used to help us make our recommendations for the best performance on the Analysis Qual, but we would like you to indicate which four problems you are submitting as representing your best effort.

Part I

1. Let A and B be two subsets of $[0,1]$ whose union is all of $[0,1]$. Show that $m^*(A) \geq 1 - m^*(B)$. (Here, m^* denotes Lebesgue outer measure.)

2. Define the following function on $C[0,1] \times C[0,1]$: $d(f,g) = \int_0^1 |f(x) - g(x)|\, dx$. Show that d is a metric on $C[0,1]$ *and* determine whether $C[0,1]$ is complete in this metric.

3. Let A be a subset of $\mathbb{R}$ with the property that for each $\epsilon > 0$ there are (Lebesgue) measurable sets B and C such that
$$B \subset A \subset C$$
and $m(C \cap B^c) < \epsilon$. Show that A is measurable. (Here, m denotes Lebesgue measure.)

4. True-false: Let f be a non-negative continuous function on $\mathbb{R}$ and suppose $\int_{\mathbb{R}} f(x)\, dx < \infty$, then $\lim_{|x|\to\infty} f(x) = 0$.

5. True-False: If $\{f_n\}_{n\geq 1}$ is a sequence of (Lebesgue) measurable functions such that
$$0 \leq f_1 \leq f_2 \leq f_3 \leq \cdots,$$
if $\sup \int_{\mathbb{R}} f_n(x)\, dx < \infty$ and if $f(x) = \lim f_n(x)$ for all x, then $\{x \mid f(x) = \infty\}$ has measure zero.

Part II

1. Calculate the radius of convergence of the power series
$$\sum_{n=0}^{\infty} z^{n^2}.$$

2. Suppose f is analytic in the region $0 < |z| < 1$ and suppose there is a constant K such that
$$|f(z)| \leq K|z|^{-\frac{1}{2}}$$
there. What kind of isolated singularity does f have at zero? (Please prove your answer.)

3. For what values of z does the series
$$\sum_{n=0}^{\infty} \frac{z^n}{1-z^n}$$
converge? Is the sum an analytic function of z?

4. True-false: There is a non-constant entire function f such $f(z+1) = f(z)$ and $f(z+i) = f(z)$ for all z in $\mathbb{C}$.

5. True-false: Suppose $\{f_n\}_{n=0}^{\infty}$ is a sequence of functions defined and analytic in the open unit disc, $|z| < 1$. Suppose also that the values of each f_n are contained in the upper half-plane, i.e., suppose that $\mathrm{Im}(f_n(z)) \geq 0$ for all n and for all z, $|z| < 1$. Then $\{f_n\}_{n=0}^{\infty}$ is a normal family.