MS EXAM ON NUMERICAL ANALYSIS

Directions: Answer all the 6 problems.

1. Give the Newton method for solving the equation

\[ e^x + 4e^{-x} - 4 = 0, \]

and discuss the convergence order of the method.
2. Let \( f(x) = \sin(\pi x) \). Determine a function \( p(x) \) such that \( p(x) \) is a polynomial on \([0, 0.5] \) and \([0.5, 1] \), and satisfies the conditions
\[
p(x) = f(x), \quad p'(x) = f'(x), \quad \text{for } x = 0, 0.5, 1.
\]
3. (a) Find $A_0$, $A_1$ and $A_2$ such that the integration rule

$$I(f) = \int_{-h}^{h} f(x) \, dx \approx A_0 f(-h/2) + A_1 f(0) + A_2 f(h/2)$$

is exact for polynomials of degree $\leq 2$.

(b) Show that the rule constructed in (a) is in fact exact for polynomials of degree $\leq 3$.

(c) For the constructed rule, it can be proved that

$$I(f) - (A_0 f(-h/2) + A_1 f(0) + A_2 f(h/2)) = c_0 f^{(4)}(\eta) h^5, \quad \eta \in [-h, h]$$

where $c_0$ is a constant independent of $f$. Find the constant $c_0$. 

4. Suppose that $B \approx A^{-1}$. Starting with an initial guess $x_0$, consider the following residual correction method:

\[
\text{for } k = 0, 1, 2, \ldots \\
r_k \leftarrow Ax_k - b; \\
x_{k+1} \leftarrow x_k - Br_k.
\]

Show that this will converge (using exact arithmetic) so that $x_k \to x$, with $x$ the exact solution, provided $\|I - BA\| < 1$. 

5. What is the QR factorization of a matrix? Explain how a QR factorization of a matrix can be computed using any of the following three methods: (i) Gram–Schmidt orthogonalization, (ii) Givens’ rotations, or (iii) Householder reflectors.

An overdetermined linear system $Ax = b$ where $A$ is $m \times n$ with $m > n$ usually cannot be solved for $x$. Instead we can ask to minimize $\|Ax - b\|_2$ over all $x$. Show how the solution to this minimization problem can be computed using the QR factorization of $A$. 

6. Consider the following two methods for numerically solving an initial value problem for the ODE $dx/dt = f(t, x)$:

\[
\begin{align*}
    x_{n+1} &= x_{n-1} + 2h f(t_n, x_n) \quad \text{(leap-frog method)} \\
    x_{n+1} &= x_n + \left(\frac{h}{2}\right)[f(t_n, x_n) + f(t_{n+1}, x_{n+1})] \quad \text{(implicit mid-point method)}
\end{align*}
\]

where $h$ is the step size, and $t_k = t_0 + k h$. For the test equation $dx/dt = \lambda x$, show that the leap-frog method is only stable for $\lambda h = 0$, while the implicit mid-point method is stable for all $\lambda h < 0$. 