MS comprehensive examination on Differential Equations/ODEs. Wednesday April 5, 2006.

Answer 4 out of 7 questions. Show your calculations and justify your answers.

1. The origin $(0,0)^T$ is a fixed point of the following autonomous system of nonlinear ODEs

$$\dot{x}_1 = -x_1^3 + x_1 x_2^2, \quad \dot{x}_2 = -2x_1^2 x_2 - x_2^3.$$

   (a) Is the origin $(0,0)^T$ a hyperbolic fixed point?

   (b) Is the function $L(x_1, x_2) := x_1^2 + x_2^2$ a (strict) Lyapunov function for this system of ODEs at $(0,0)^T$?

   (c) Is the constant solution $x(t) = (0,0)^T$ asymptotically Lyapunov-stable?

2. Consider the autonomous nonlinear system of ODEs given by

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_1 - x_2 - x_1(4x_1^2 + x_2^2) \\ x_1 + x_2 - x_2(x_1^2 + x_2^2) \end{pmatrix}.$$

   In order to show that there is a periodic solution in the annular region

$$D = \left\{ x \in \mathbb{R}^2 \mid \frac{1}{2} \leq \|x\|_2 \leq 1 \right\}$$

   by application of the Poincaré-Bendixson theorem, follow those steps:

   (a) Convert the nonlinear system of ODEs given in Cartesian coordinates (x_1, x_2) to polar coordinates (r, θ). Hint: remember that these coordinates are related by $x_1 = r\cos(\theta), x_2 = r\sin(\theta)$ and they satisfy the relations $r\dot{r} = x_1\dot{x}_1 + x_2\dot{x}_2$, $r^2\dot{\theta} = x_1\dot{x}_2 - \dot{x}_1 x_2$.

   (b) Show that all trajectories do not have a radially inward component on the circle of radius $r_1 = 1/2$ centered at the origin $(0,0)^T$. Show also that all trajectories do not have a radially outward component on the circle of radius $r_2 = 1$ centered at the origin $(0,0)^T$.

   (c) Verify the conditions of the Poincaré-Bendixson theorem to conclude that the system of ODEs has a periodic orbit in the annular region D.

3. Consider the following autonomous system of linear ODEs

$$\dot{x}_1 = -x_1 + 2x_2, \quad \dot{x}_2 = 2x_1 - x_2.$$

   (a) Find its general solution.
   (b) Classify the fixed point at the origin $(0,0)^T$.
   (c) Sketch the phase portrait.

4. We consider the autonomous nonlinear system of ODEs given by

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -x_1 - x_2 + x_1^2 + x_2^2 \end{pmatrix}.$$

   Does this system have a periodic orbit or not? Hint: Apply Dulac's criterion with $\Psi(x_1, x_2) := e^{-2x_1}$.

5. For the following one-dimensional autonomous nonlinear ODE, find all the fixed points as a function of the parameter $\mu \in \mathbb{R}$, determine their Lyapunov-stability, sketch a bifurcation diagram, and find the bifurcation value(s) of the parameter $\mu \in \mathbb{R}$:

$$\dot{x} = \mu^3 x - \mu x^3.$$

6. Consider the following autonomous system of differential equations

$$\dot{x}_1 = x_1(2 - x_1 - x_2), \quad \dot{x}_2 = x_2(2 - 4x_1 - x_2).$$

   (a) Find all fixed points.
   (b) Which of these fixed points are hyperbolic? If a fixed point is hyperbolic say if it is a source, a sink, or a saddle.

7. Consider the system of ODEs $dy/dt = f(t, y)$ and the following explicit Runge-Kutta method called Heun's method

$$\begin{aligned} Y_1 &= y_0 \\ Y_2 &= y_0 + h\frac{1}{3} f(t_0, Y_1) \\ Y_3 &= y_0 + h\frac{2}{3} f(t_0 + h/3, Y_2) \\ y_1 &= y_0 + h\left(\frac{1}{4} f(t_0, Y_1) + \frac{3}{4} f(t_0 + 2h/3, Y_3)\right) \end{aligned}$$

   (a) What is the stability function $R(z)$ of this method ($z := h\lambda$ and $y' = \lambda y$)?
   (b) What is the local order of this method?