MS comprehensive examination on Differential Equations/ODEs. Wednesday April 5, 2006.

Answer 4 out of 7 questions. Show your calculations and justify your answers.

1. The origin \((0,0)^T\) is a fixed point of the following autonomous system of nonlinear ODEs

\[
\begin{align*}
\dot{x}_1 &= -x_1^3 + x_1 x_2^2, \\
\dot{x}_2 &= -2x_1^2 x_2 - x_2^3.
\end{align*}
\]

(a) Is the origin \((0,0)^T\) a hyperbolic fixed point?

(b) Is the function \(L(x_1,x_2) := x_1^2 + x_2^2\) a (strict) Lyapunov function for this system of ODEs at \((0,0)^T\)?

(c) Is the constant solution \(x(t) = (0,0)^T\) asymptotically Lyapunov-stable?

2. Consider the autonomous nonlinear system of ODEs given by

\[
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{pmatrix} = \begin{pmatrix}
x_1 - x_2 - x_1(4x_1^2 + x_2^2) \\
x_1 + x_2 - x_2(x_1^2 + x_2^2)
\end{pmatrix}.
\]

In order to show that there is a periodic solution in the annular region

\[
D = \left\{ x \in \mathbb{R}^2 \mid \frac{1}{2} \leq \|x\|_2 \leq 1 \right\}
\]

by application of the Poincaré-Bendixson theorem, follow those steps:

(a) Convert the nonlinear system of ODEs given in Cartesian coordinates \((x_1,x_2)\) to polar coordinates \((r,\theta)\). Hint: remember that these coordinates are related by \(x_1 = r \cos(\theta), x_2 = r \sin(\theta)\) and they satisfy the relations \(r\dot{r} = x_1 \dot{x}_1 + x_2 \dot{x}_2, r^2 \dot{\theta} = x_1 \dot{x}_2 - x_2 \dot{x}_1\).

(b) Show that all trajectories do not have a radially inward component on the circle of radius \(r_1 = 1/2\) centered at the origin \((0,0)^T\). Show also that all trajectories do not have a radially outward component on the circle of radius \(r_2 = 1\) centered at the origin \((0,0)^T\).

(c) Verify the conditions of the Poincaré-Bendixson theorem to conclude that the system of ODEs has a periodic orbit in the annular region \(D\).
3. Consider the following autonomous system of linear ODEs
\[
\begin{align*}
\dot{x}_1 &= -x_1 + 2x_2, \\
\dot{x}_2 &= 2x_1 - x_2.
\end{align*}
\]
(a) Find its general solution.
(b) Classify the fixed point at the origin \((0,0)^T\).
(c) Sketch the phase portrait.

4. We consider the autonomous nonlinear system of ODEs given by
\[
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{pmatrix} = \begin{pmatrix}
x_2 \\
-x_1 - x_2 + x_1^2 + x_2^2
\end{pmatrix}.
\]
Does this system have a periodic orbit or not? Hint: Apply Dulac’s criterion with \(\Psi(x_1, x_2) := e^{-2x_1}\).

5. For the following one-dimensional autonomous nonlinear ODE, find all the fixed points as a function of the parameter \(\mu \in \mathbb{R}\), determine their Lyapunov-stability, sketch a bifurcation diagram, and find the bifurcation value(s) of the parameter \(\mu \in \mathbb{R}\):
\[
\dot{x} = \mu^3 x - \mu x^3.
\]

6. Consider the following autonomous system of differential equations
\[
\begin{align*}
\dot{x}_1 &= x_1(2 - x_1 - x_2), \\
\dot{x}_2 &= x_2(2 - 4x_1 - x_2).
\end{align*}
\]
(a) Find all fixed points.
(b) Which of these fixed points are hyperbolic? If a fixed point is hyperbolic say if it is a source, a sink, or a saddle.

7. Consider the system of ODEs \(dy/dt = f(t,y)\) and the following explicit Runge-Kutta method called Heun’s method
\[
\begin{align*}
Y_1 &= y_0 \\
Y_2 &= y_0 + h \frac{1}{3} f(t_0, Y_1) \\
Y_3 &= y_0 + h \frac{2}{3} f(t_0 + h/3, Y_2) \\
y_1 &= y_0 + h \left( \frac{1}{4} f(t_0, Y_1) + \frac{3}{4} f(t_0 + 2h/3, Y_3) \right)
\end{align*}
\]
(a) What is the stability function \(R(z)\) of this method (\(z := h\lambda\) and \(y' = \lambda y\))? 
(b) What is the local order of this method?