(Solve three of the problems.)

1. Solve the initial value problem

\[ u_t + xu_x = u \]

with \( u(x,0) = x^2 \). Describe and draw the characteristics.

2. Using separation of variables, find the eigenfunctions of the Laplace operator with Dirichlet boundary conditions on the rectangle \([0, \pi] \times [0, 2\pi]\).

3. Assume \( u \) is twice continuously differentiable on \([0,1]^n \subset \mathbb{R}^n\), that \( u \) is zero on the boundary of that domain and \(|\Delta u| \leq 1\). Use the maximum principle to give an estimate of the size of the function \( u \).

4. What is the proper weak solution (i.e. the solution fulfilling the Lax entropy condition) of the equation

\[ u_t + u^3 \cdot u_x = 0 \]

for the initial values

\[ f_1(x) = \begin{cases} \frac{1}{2} & \text{for } x > 0 \\ -2 & \text{for } x \leq 0 \end{cases} \]

and

\[ f_2(x) = \begin{cases} \frac{1}{2} & \text{for } x > 0 \\ -2 & \text{for } x \leq 0 \end{cases} \]

5. Compute the Fourier series

\[ \sum_{k=0}^{\infty} a_k \cos(kx) \]

for the function

\[ f(x) = \begin{cases} 1 & \text{for } x \in [0, \pi/2] \\ -1 & \text{for } x \in (\pi/2, \pi] \end{cases} \]
on the interval \([0, \pi]\). Also solve the heat equation \(u_t(x, t) = u_{xx}(x, t)\) on the square \([0, \pi] \times [0, \infty)\) with the initial value \(u(x, 0) = f(x)\) and the boundary condition \(u_x(0, t) = u_x(\pi, t) = 0\). What does it converge to as \(t \to \infty\)?