Qualifying Examination Fall 2006, Section on Partial Differential Equations

August 23, 2006

(Solve three of the problems.)

1. Solve the initial value problem

   \[ u_t + 3t^2 u_x = u \]

   with \( u(x, 0) = x^2 \). Describe and draw the characteristics.

2. Compute the Fourier series

   \[ \sum_{k=0}^{\infty} a_k \cos(kx) \]

   for the function

   \[
   f(x) = \begin{cases} 
   1 & \text{for } x \in [0, \pi/2] \\
   0 & \text{for } x \in (\pi/2, \pi]
   \end{cases}
   \]

   on the interval \([0, \pi]\). Also solve the heat equation \( u_t(x, t) = u_{xx}(x, t) \) on the square \([0, \pi] \times [0, \infty)\) with the initial value \( u(x, 0) = f(x) \) and the boundary condition \( u_x(0, t) = u_x(\pi, t) = 0 \). What does it converge to as \( t \to \infty \) ?

3. What is the proper weak solution (i.e. the solution fulfilling the Lax entropy condition) of the equation

   \[ u_t + u^9 \cdot u_x = 0 \]

   for the initial values

   \[
   f_1(x) = \begin{cases} 
   1 & \text{for } x > 0 \\
   0 & \text{for } x \leq 0
   \end{cases}
   \]

   and

   \[
   f_2(x) = \begin{cases} 
   1 & \text{for } x > 0 \\
   -1 & \text{for } x \leq 0
   \end{cases}
   \]

4. Assume \( u \in C^2 \cap \mathbb{R}^3 \mid |x| \leq 1 \), \( \Delta u \leq 6 \) and \( u(x) \geq 0 \) for \( |x| = 1 \). How small can \( u(0) \) become ?

5. Using separation of variables, find the eigenfunctions of the Laplace operator with Neumann boundary conditions on the rectangle \([0, 3\pi] \times [0, \pi]\).