Qualifying Exam: PDE, Fall, 2007

Choose any three out of the six problems.

1. Solve the initial value problem

$u_t + xu_x = 0, \quad x \in R, \ t > 0, \ u(x,0) = f(x), \quad x \in R$

where $f(x) \in C^1(R)$. Over what region in the x-t plane does the solution exist? Draw the characteristics on the x-t plane where the solution exists.

2. Find a weak solution satisfying the entropy conditions for

$u_t + (u^3)_x = 0, \quad x \in R, \ t > 0,$

with initial data $u(x,0) = \begin{cases} 2 & x < 0 \\ 1 & x \geq 0 \end{cases}$

and with initial data $u(x,0) = \begin{cases} 1 & x < 0 \\ 2 & x \geq 0 \end{cases}$.

(ii) Write an upwind scheme for the above problem. What is the CFL condition for the scheme?

3. Solve the following problem

$u_t - u_{xx} = 0, \quad x > 0, \ t > 0,$

$u(x,0) = f(x), \quad x \geq 0, \ u(0,t) = 2, \quad t \geq 0$

where $f(0) = 2$. Given that $f(x)$ is continuous and $|f(x)| \leq M$ for $x \geq 0$, find the maximum of $|u(x,t)|$ for $x \geq 0$ and $t \geq 0$.

4. Compute the Fourier series

$\Sigma_{k=0}^{+\infty} a_k \cos(kx)$

for function

$f(x) = \begin{cases} 2 & x \in [0, \frac{\pi}{2}] \\ 0 & x \in (\frac{\pi}{2}, \pi] \end{cases}$

on the interval $[0, \pi]$. Also solve the heat equation $u_t = u_{xx}$ on $[0,\pi] \times [0, +\infty)$ with the initial value $u(x,0) = f(x)$ and the boundary conditions $u_x(0,t) = u_x(\pi,t) = 0$. What does the solution converge to as $t \longrightarrow +\infty$?

5. Solve the initial-boundary-value problem for the wave equation on the half-line

$$u_{tt} - c^2 u_{xx} = 0, \quad x > 0, \ t > 0,$$
$$u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad x \geq 0,$$
$$u_x(0,t) = 0, \quad t > 0.$$

6. Consider the wave equation with damping

$$u_{tt} + du_t - c^2 u_{xx} = 0, \quad x \in R, \ t > 0,$$
$$u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad x \in R.$$

where $d > 0$, f and g are smooth functions with compact support. Show that the energy $e(t)$ decays as t increases.