Qualifying Exam: PDE, Fall, 2008

Choose any three out of the six problems.

1. (i) Solve the initial value problem for the linear equation
   \[ u_t + (x^2 + 1)u_x = 0, \quad x \in \mathbb{R}, \quad t > 0, \quad u(x, 0) = x^2, \quad x \in \mathbb{R}. \]
   (ii) Over what region in the \( x-t \) plane does the solution exist? Draw the characteristics on the \( x-t \) plane where the solution exists.

2. (i) Find the bounded solution \( u \) to the following initial-boundary-value problem
   \[ u_t - u_{xx} = 0, \quad x > 0, \quad t > 0, \]
   \[ u(x, 0) = f(x), \quad x \geq 0, \quad u(0, t) = 2, \quad t \geq 0 \]
   where \( f \) is continuous on \([0, +\infty)\) satisfying \( f(0) = 2 \) and \( \sup_{x \geq 0} |f(x)| = M < +\infty \).
   (ii) Find the supremum of \( |u(x, t)| \) for \( x \geq 0 \) and \( t \geq 0 \) in terms of the given data.

3. Compute the Fourier series
   \[ \sum_{k=0}^{+\infty} a_k \cos(kx) \]
   for function
   \[ f(x) = \begin{cases} 1 & x \in [0, \frac{\pi}{2}] \\ 0 & x \in (\frac{\pi}{2}, \pi] \end{cases} \]
   on the interval \([0, \pi]\). Also solve the heat equation \( u_t = u_{xx} \) on \([0, \pi] \times [0, +\infty)\) with the initial value \( u(x, 0) = f(x) \) and the boundary conditions \( u_x(0, t) = u_x(\pi, t) = 0 \). What does the solution converge to as \( t \to +\infty \)？

4. Solve the following initial-boundary-value problem
   \[ u_{tt} - u_{xx} = 0, \quad x > 0, \quad t > 0, \]
   \[ u(x, 0) = f(x), \quad u_t(x, 0) = g(x), \quad x \geq 0, \]
   \[ u(0, t) = 0, \quad t > 0 \]
   where \( f \) and \( g \) are smooth functions satisfying \( f(0) = g(0) = 0 \).

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5. (i) Find a weak solution satisfying the entropy conditions for 
\[ u_t + \left(\frac{u^2}{2}\right)_x = 0, \quad x \in \mathbb{R}, \quad t > 0, \]
with initial data 
\[ u(x, 0) = \begin{cases} 
2 & x < 0 \\
1 & x \geq 0 
\end{cases} \]
and with initial data 
\[ u(x, 0) = \begin{cases} 
1 & x < 0 \\
2 & x \geq 0 
\end{cases} . \]

(ii) Write an upwind scheme for the above problems. What is the CFL condition for the scheme?

6. Consider the wave equation problem 
\[ u_{tt} - c^2 u_{xx} = q(x, t), \quad x \in \mathbb{R}, \quad t > 0, \]
\[ u(x, 0) = 0, \quad u_t(x, 0) = 0, \quad x \in \mathbb{R} \]
where \( c > 0 \) and 
\[ q(x, t) = \begin{cases} 
(1 - x^2) \sin t & |x| \leq 1 \\
0 & |x| > 1 
\end{cases} . \]
Show that \( u(x, t) = 0 \) for \( |x| > ct + 1 \).