Qualifying Exam: PDE, Fall, 2008

Choose any three out of the six problems.

1. (i) Solve the initial value problem for the linear equation

$u_t + (x^2+1)u_x = 0, \;\; x \in R, \; t > 0, \; u(x,0) = x^2, \;\; x \in R.$

(ii) Over what region in the x-t plane does the solution exist? Draw the characteristics on the x-t plane where the solution exists.

2. (i) Find the bounded solution u to the following initial-boundary-value problem

$u_t - u_{xx} = 0, \;\; x > 0, \; t > 0,$

$u(x,0) = f(x), \;\; x \geq 0, \; u(0,t) = 2, \;\; t \geq 0$

where f is continuous on $[0,+\infty)$ satisfying $f(0) = 2$ and $\sup_{x\geq 0}|f(x)| = M < +\infty$.

(ii) Find the supremum of $|u(x,t)|$ for $x \geq 0$ and $t \geq 0$ in terms of the given data.

3. Compute the Fourier series

$\Sigma_{k=0}^{+\infty} a_k \cos(kx)$

for function

$$f(x) = \begin{cases} 1 & x \in [0, \frac{\pi}{2}] \\ 0 & x \in (\frac{\pi}{2}, \pi] \end{cases}$$

on the interval $[0,\pi]$. Also solve the heat equation $u_t = u_{xx}$ on $[0,\pi]\times[0,+\infty)$ with the initial value $u(x,0) = f(x)$ and the boundary conditions $u_x(0,t) = u_x(\pi,t) = 0$. What does the solution converge to as $t \to +\infty$?

4. Solve the following initial-boundary-value problem

$u_{tt} - u_{xx} = 0, \;\; x > 0, \; t > 0,$

$u(x,0) = f(x), \;\; u_t(x,0) = g(x), \;\; x \geq 0,$

$u(0,t) = 0, \;\; t > 0$

where f and g are smooth functions satisfying $f(0) = g(0) = 0$.

5. (i) Find a weak solution satisfying the entropy conditions for

$$u_t + (\frac{u^2}{2})_x = 0, \quad x \in R, \ t > 0,$$

with initial data $u(x,0) = \begin{cases} 2 & x < 0 \\ 1 & x \geq 0 \end{cases}$

and with initial data $u(x,0) = \begin{cases} 1 & x < 0 \\ 2 & x \geq 0 \end{cases}$.

(ii) Write an upwind scheme for the above problems. What is the CFL condition for the scheme?

6. Consider the wave equation problem

$$u_{tt} - c^2 u_{xx} = q(x,t), \quad x \in R, \ t > 0,$$

$$u(x,0) = 0, \quad u_t(x,0) = 0, \quad x \in R$$

where $c > 0$ and

$$q(x,t) = \begin{cases} (1 - x^2)\sin t & |x| \leq 1 \\ 0 & |x| > 1 \end{cases}.$$

Show that $u(x,t) = 0$ for $|x| > ct + 1$.