Qualifying Exam: PDE, Spring, 2008

Choose any three out of the six problems.

1. (i) Find a weak solution satisfying the entropy conditions for

$$u_t + (2u^2)_x = 0, \quad x \in R, \ t > 0,$$

with initial data $u(x,0) = \begin{cases} 2 & x < 0 \\ 1 & x \geq 0 \end{cases}$

and with initial data $u(x,0) = \begin{cases} 1 & x < 0 \\ 2 & x \geq 0 \end{cases}$.

(ii) Write an upwind scheme for the above problem. What is the CFL condition for the scheme?

2. (i) Find the bounded solution u to the following initial-boundary-value problem

$$u_t - u_{xx} = 0, \quad x > 0, \ t > 0,$$
$$u(x,0) = f(x), \quad x \geq 0, \ u(0,t) = 1, \quad t \geq 0$$

where f is continuous on $[0, +\infty)$ satisfying $f(0) = 1$ and $|f(x)| \leq M$ for $x \geq 0$, where $M > 0$.

(ii) Find the supremum of $|u(x,t)|$ for $x \geq 0$ and $t \geq 0$ in terms of the given data.

3. Compute the Fourier series

$$\Sigma_{k=1}^{+\infty} b_k \sin(kx)$$

for function

$$f(x) = \begin{cases} 1 & x \in [0, \frac{\pi}{2}] \\ 0 & x \in (\frac{\pi}{2}, \pi] \end{cases}$$

on the interval $[0, \pi]$. Also solve the heat equation $u_t = u_{xx}$ on $[0,\pi] \times [0, +\infty)$ with the initial value $u(x,0) = f(x)$ and the boundary conditions $u(0,t) = u(\pi,t) = 0$. What does the solution converge to as $t \to +\infty$?

4. Solve the following initial-boundary-value problem

$$u_{tt} - u_{xx} = 0, \quad x > 0, \ t > 0,$$
$$u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad x \geq 0,$$
$$u_x(0,t) = 0, \quad t > 0$$

where f and g are smooth functions satisfying $f'(0) = 0$ and $g'(0) = 0$.

5. Solve the initial value problem

$$u_t + 2xu_x = 0, \quad x \in R,\ t > 0,\ u(x,0) = f(x), \quad x \in R$$

where $f(x) \in C^1(R)$. Over what region in the x-t plane does the solution exist? Draw the characteristics on the x-t plane where the solution exists.

6. Consider the wave equation with damping

$$u_{tt} + du_t - c^2 u_{xx} = 0, \quad x \in R,\ t > 0,$$

$$u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad x \in R.$$

where $c > 0$, $d > 0$, and f, g are smooth functions with compact support. Define the energy $e(t)$ and show that the energy decays as t increases.