"Harmonic Analysis: Smooth and Non-smooth."

Introduction to the Conference: Historical context, Reading list, Basic definitions, some relevant results, and an Outline of the Ten Lectures by Jorgensen.

CBMS conference to be held at Iowa State University,

June 4–8, 2018.

Speaker: Palle Jorgensen, Professor, University of Iowa.
Organizers: Professors John Herr, Justin Peters, and Eric Weber.

Lecture 1. Harmonic Analysis of Measures: Analysis on Fractals

Lecture 2. Spectra of measures, tilings, and wandering vectors

Lecture 3. The universal tiling conjecture in dimension one and operator fractals

Lecture 4. Representations of Cuntz algebras associated to quasi-stationary Markov measures

Lecture 5. The Cuntz relations and kernel decompositions

Lecture 6. Harmonic analysis of wavelet filters: input-output and state-space models

Lecture 7. Spectral theory for Gaussian processes: reproducing kernels, boundaries, and L^2-wavelet generators with fractional scales

Lecture 8. Reproducing Kernel Hilbert Spaces arising from groups

Lecture 9. Extensions of positive definite functions

Lecture 10. Reflection positive stochastic processes indexed by Lie groups

Background There is a recent and increasing interest in understanding the harmonic analysis of non-smooth geometries, typically fractal like. They are unlike the familiar smooth Euclidean geometry. In the non-smooth case, nearby points are not locally connected to each other. Real-world examples where these types of geometry appear include large computer networks, relationships in datasets, and fractal structures such as those found in crystalline substances, light scattering, and other natural phenomena where dynamical systems are present.

Details and Format A series of ten lectures by Professor Palle Jorgensen from the University of Iowa, a leader in the fields of smooth and non-smooth harmonic analysis. The conference aims to demonstrate surprising connections between the two domains of geometry and Fourier spectra. In addition to the 10 lectures by Jorgensen, there will be other invited speakers.

The conference aims to bring both experienced and new researchers together to stimulate collaboration on this timely topic. It also aims to advance representation and participation of underrepresented minorities within mathematics, and the development of a globally competitive STEM workforce.

There is NSF support, including to current graduate students. The conference will contribute to those graduate students' educational and professional development and hence prepare the nation's next generation of researchers to engage this increasingly important subject area.

Jorgensen and Pedersen showed in 1998 that there exists a Cantor-like set with the property that the uniform measure supported on that set is spectral, meaning that there exists a sequence of frequencies for which the corresponding exponential functions form an orthonormal basis in the Hilbert space of square-integrable functions with respect to that measure. Research that has been inspired by this stunning result includes: fractal Fourier analyses, spectral theory of Ruelle transfer operators, representation theory of Cuntz algebras, convergence of the cascade algorithm in wavelet theory, reproducing kernels and their boundary representations, Bernoulli convolutions, and Markov processes. The remarkable feature of this array of subjects is that they straddle both the smooth and non-smooth settings. The lectures presented by Professor Jorgensen will unify these far-reaching research areas at the interface of smooth and non-smooth harmonic analysis.

Preview

Smooth harmonic analysis refers to harmonic analysis over a connected or locally connected domain–typically Euclidean space or locally connected subsets of Euclidean space. The classical example of this is the existence of Fourier series expansions for square integrable functions on the unit interval. Non-smooth harmonic analysis then refers to harmonic analysis on discrete or disconnected domains– typical examples of this setting are Cantor like subsets of the real line and analogous fractals in higher dimensions. In 1998, Jorgensen and Steen Pedersen proved a remarkable result: there exists a Cantor like set (of Hausdorff dimension $1/2$) with the property that the uniform measure supported on that set is spectral, meaning that there exists a sequence of frequencies for which the exponentials form an orthonormal basis in the Hilbert space of square integrable functions with respect to that measure. This surprising result, together with results of Robert Strichartz, has lead to a plethora of new research directions in non-smooth harmonic analysis.

Research that has been inspired by this surprising result includes: fractal Fourier analyses (fractals in the large), spectral theory of Ruelle operators; representation theory of Cuntz algebras; convergence of the cascade algorithm in wavelet theory; reproducing kernels and their boundary representations; Bernoulli convolutions and Markov processes. The remarkable aspect of these broad connections is that they often straddle both the smooth and non-smooth domains. This is particularly evident in Jorgensen's research on the cascade algorithm, as wavelets already possess a "dual" existence in the continuous and discrete worlds, and also his research on the boundary representations of reproducing kernels, as the non-smooth domains appear as boundaries of smooth domains. In work with Dorin Dutkay, Jorgensen showed that the general affine IFS-systems, even if not amenable to Fourier analysis, in fact do admit wavelet bases, and so in particular can be analyzed with the use of multiresolutions. In recent work with Herr and Weber, Jorgensen has shown that fractals that are not spectral (and so do not admit an orthogonal Fourier analysis) still admits a harmonic analysis as boundary values for certain subspaces of the Hardy space of the disc and the corresponding reproducing kernels within them.

The lectures to be given by Jorgensen will cover the following overarching themes: the harmonic analysis of Cantor spaces (and measures) arising as fractals (including fractal dust) and iterated function systems (IFSs), as well as the methods used to study their harmonic analyses that span both the smooth and non-smooth domains. A consequence of the fact that these methods form a bridge between the smooth and non-smooth domain is that the topics to be discussed– while on the surface seem largely unrelated–actually are closely related and together form a tightly focused theme. The breadth of topics will attract a broader audience of established researchers, while the interconnectedness and sharply focused nature of these topics will prove beneficial to beginning researchers in non-smooth harmonic analysis.

Introduction

One of the most fruitful achievements of mathematics in the past five hundred years has been the development of Fourier series. Such a series may be thought of as the decomposition of a periodic function into sinusoid waves of varying frequencies. Application of such decompositions are naturally abundant, with waves occurring in all manner of physics, and uses for periodic functions being present in other areas such as economics and signal processing, just to name a few. The importance of Fourier series is well-known and incontestable.

Historical context While to many non-mathematicians and undergraduate math majors, a Fourier series is regarded as a breakdown into sine and cosine waves, the experienced analyst will usually think of it (equivalently), as a decomposition into sums of complex exponentials. For instance, in the classical setting of the unit interval $[0,1)$, a Lebesgue integrable function $f:[0,1)\to\mathbb{C}$ will induce a Fourier series

$$f(x) \sim \sum\nolimits_{n\in\mathbb{Z}} \hat{f}(n)\, e^{i2\pi nx} \tag{1}$$

where

$$\hat{f}(n) := \int_0^1 f(x)\, e^{-i2\pi nx} dx. \tag{2}$$

Because the Fourier series is intended to represent the function $f(x)$, it is only natural to ask in what senses, if any, the sum above converges to $f(x)$. One can ask important questions about pointwise convergence, but it is more relevant for our purposes to restrict attention to various normed spaces of functions or, as we will be most concerned with hereafter, a Hilbert space consisting of square-integrable functions, and then ask about norm convergence. In our present context, if we let $L^2([0,1))$ denote the Hilbert space of (equivalence classes of) functions $f:[0,1)\to\mathbb{C}$ satisfying

$$\|f\|^2 := \int_0^1 |f(x)|^2\,dx < \infty \tag{3}$$

and equipped with the inner product

$$\langle f,g\rangle := \int_0^1 f(x)\overline{g(x)}dx, \tag{4}$$

then if $f\in L^2([0,1))$, the convergence in (1) will occur in the norm of $L^2([0,1))$. It is also easy to see that in $L^2([0,1))$,

$$\left\langle e^{i2\pi mx}, e^{i2\pi nx}\right\rangle = \int_0^1 e^{i2\pi mx}e^{-i2\pi mx}dx = \begin{cases}1 & \text{if } m=n\\ 0 & \text{otherwise.}\end{cases} \tag{5}$$

That is, the set of complex exponentials $\left\{e^{i2\pi nx}\right\}_{n\in\mathbb{Z}}$ is orthogonal in $L^2([0,1))$. Since every function in $L^2([0,1))$ can be written in terms of these exponentials, $\left\{e^{i2\pi nx}\right\}_{n\in\mathbb{Z}}$ is in fact an orthonormal basis of $L^2([0,1))$.

Because there exists a countable set of complex exponential functions that form an orthogonal basis of $L^2([0,1))$, we say that the set $[0,1)$ is spectral. The set of frequencies of such an orthogonal basis of exponentials, which is this case is $\mathbb{Z}$, is called a spectrum.

Like most areas of analysis, the historical and most common contexts for Fourier series are also the most mundane: The functions they decompose are defined on $\mathbb{R}$, the unit interval $[0,1)$, or sometimes a discrete set. The underlying measure used for integration is Lebesgue measure. It is thanks to the work of many individuals, including Palle Jorgensen, that modern Fourier analysis has been able to aspire beyond these historical paradigms.

The first paradigm break is to consider a wider variety of domains in a wider variety of dimensions. In general, if C is a compact subset of $\mathbb{R}^n$ of nonzero Lebesgue measure, then we say that C is spectral if there exists a countable set $\Lambda\subset\mathbb{R}^n$ such that $\left\{e^{i2\pi\lambda\cdot\vec{x}}\right\}_{n\in\Lambda}$ is an orthogonal basis of $L^2(C)$, where

$$L^2(C) := \left\{ f: C\to\mathbb{C} \,\middle|\, \int_C |f(\vec{x})|^2\, d\lambda^n(\vec{x}) < \infty \right\}. \tag{6}$$

Here λ^n is Lebesgue measure in $\mathbb{R}^n$.

The famous Fuglede Conjecture surmised that C would be spectral if and only if it would tessellate by translation to cover $\mathbb{R}^n$. Iosevich, Katz, and Tao proved in 2001 that the conjecture holds for convex planar domains [IKT03]. In the same year, they also proved that a smooth, symmetric, convex body with at least one point of nonvanishing Gaussian curvature cannot be spectral [IKT01]. However, in 2003 Tao devised counterexamples to the Fuglede Conjecture in $\mathbb{R}^5$ and $\mathbb{R}^{11}$ [Tao04]. The conjecture remains open in low dimensions.

The second paradigm break is to substitute a different Borel measure in place of Lebesgue measure. For example, if μ is any Borel measure on $[0,1)$, one can form the Hilbert space

$$L^2(\mu) = \left\{ f:[0,1)\to\mathbb{C} \,\middle|\, \int_0^1 |f(x)|^2\, d\mu(x) < \infty \right\} \tag{7}$$

with inner product

$$\langle f,g\rangle_\mu = \int_0^1 f(x)\overline{g(x)}d\mu(x). \tag{8}$$

Comparing equations (7) and (8) with equations (3) and (4), respectively, we see that we can then regard spectrality as a property of measures rather than of sets: The measure μ is spectral if there exists a countable index set Λ such that the set of complex exponentials $\left\{e^{i2\pi\lambda x}\right\}_{\lambda\in\Lambda}$ is an orthogonal basis of $L^2(\mu)$. The index set Λ is then called a spectrum of μ.

Iterated function systems (IFS)

There do, of course, exist some measures that are not spectral. Of great interest to Jorgensen are measures that arise naturally from affine iterated function systems. An iterated function system (IFS) is a finite set of contraction operators $\tau_0, \tau_1, \cdots, \tau_n$ on a complete metric space S. As a consequence of Hutchinson's Theorem [Hut81], for an IFS on $\mathbb{R}^n$, there exists a unique compact set $X \subset \mathbb{R}^n$ left invariant by system in the sense that $X = \cup_{j=0}^{n} \tau_n(X)$. There will then exist a unique Borel measure μ on X such that

$$\int_X f(x)\, d\mu(x) = \frac{1}{n+1} \sum\nolimits_{j=0}^{n} \int_X f(\tau_j(x))\, d\mu(x) \tag{9}$$

for all continuous f.

In many cases of interest, X is a fractal set. In particular, if we take the iterated function system

$$\tau_0(x) = \frac{x}{3}, \quad \tau_1(x) = \frac{x+2}{3}$$

on $\mathbb{R}$, then the attractor is the ternary Cantor set C_3. The set C_3 has another construction: One starts with the interval $[0,1]$ and removes the middle third, leaving only the intervals $[0,1/3]$ and $[2/3,1]$, and then successively continues to remove the middle third of each remaining interval. Intersecting the sets remaining at each step yields C_3. The ternary Cantor measure μ_3 is then the measure induced in (9). Alternatively, μ_3 is the Hausdorff measure of dimension $\frac{\ln 2}{\ln 3}$ restricted to C_3.

In [JP98] Jorgensen and Pedersen used the zero set of the Fourier-Stieltjes transform of μ_3 to show that μ_3 is not spectral. Equally remarkably, they showed that the quaternary (4-ary) Cantor set, which is the measure induced in (9) under the IFS

$$\tau_0(x) = \frac{x}{4}, \quad \tau_1(x) = \frac{x+2}{4} \tag{10}$$

is spectral by using Hadamard matrices and a completeness argument based on the Ruelle transfer operator. The attractor set for this IFS can be described in a manner similar to the ternary Cantor set: The 4-ary set case is as follows,

$$C_4 = \left\{x \in [0,1] : x = \sum\nolimits_{k=1}^{\infty} \frac{a_k}{4^k},\ a_k \in \{0,2\}\right\},$$

and the invariant measure is denoted by μ_4. Jorgensen and Pedersen prove that

$$\begin{aligned}\Gamma_4 &= \left\{\sum\nolimits_{n=0}^{N} l_n 4^n : l_n \in \{0,1\},\ N \in \mathbb{N}\right\} \\ &= \{0,1,4,5,16,17,20,21,64,65,\cdots\}\end{aligned}$$

is a spectrum for μ_4, though there are many spectra [DHS09, DHL13]. The proof that this is a spectrum is a two step process: first the orthogonality of the exponentials with frequencies in Γ_4 is verified, and second the completeness of those exponentials are verified.

The orthogonality of the exponentials can be checked in several ways:

1. checking the zeroes of the Fourier-Stieltjes transform of μ_4;
2. using the representation of a particular Cuntz algebra on $L^2(\mu_4)$;
3. generating Γ_4 as the invariant set for a second IFS that is "dual" in a sense to the IFS in (10) ("fractals in the large").

While these three methods are distinct, they all rely on the fact that a certain matrix associated to the IFSs is a (complex) Hadamard matrix. All three of these methods are, more or less, contained in the original paper [JP98].

As a Borel probability measure, μ_4 is determined uniquely by the following IFS-fixed-point property:

$$\mu_4 = \frac{1}{2}\left(\mu_4 \circ \tau_0^{-1} + \mu_4 \circ \tau_1^{-1}\right),$$

see (10) for the affine maps τ_i, $i = 0, 1$; and one checks that the support of μ_4 is the 4-ary Cantor set C_4.

By contrast, when this is modified to (μ_3, C_3), the middle-third Cantor, Jorgensen and Pedersen proved that then there cannot be more than two orthogonal Fourier functions $e_\lambda(x) = e^{i2\pi\lambda x}$, for any choices of points λ in $\mathbb{R}$.

The completeness of the exponentials (for the cases when the specified Cantor measure is spectral) can be shown in several ways as well, though the completeness is more subtle. The original argument for completeness given in [JP98] uses a delicate analysis of the spectral theory of a Ruelle transfer operator. Jorgensen and Pedersen construct an operator on $C(\mathbb{R})$ using filters associated to the IFS in (10), which they term a Ruelle transfer operator. The argument then is to check that the eigenvalue 1 for this operator is a simple eigenvalue. An alternative argument for completeness given by Strichartz in [Str98] uses the convergence of the cascade algorithm from wavelet theory [Mal89, Dau88, Law91]. Later arguments for completeness were developed in [DJ09, DJ12b] again using the representation theory of Cuntz algebras.

The Cuntz algebra $\mathscr{O}_N$ for $N \geq 2$ is the universal C^*-algebra generated by a family $\{S_0, \cdots, S_{N-1}\}$ of N isometries satisfying the relation

$$\sum\nolimits_{j=0}^{N-1} S_j S_j^* = I, \quad \text{and} \quad S_i^* S_j = \delta_{ij} I.$$

There are many ways to generate such families. For example, consider the isometries S_0, S_1 on $L^2[0,1]$ given by defining their adjoints

$$(S_0^* f)(x) = \frac{1}{\sqrt{2}} f\left(\frac{x}{2}\right) \quad \text{and} \quad (S_1^* f)(x) = \frac{1}{\sqrt{2}} f\left(\frac{x+1}{2}\right),$$

$f \in L^2[0,1]$, $x \in [0,1]$. One can check that the range isometries $S_0 S_0^* = \chi_{[0,1/2]}$ and $S_1 S_1^* = \chi_{[1/2,1]}$, so that the Cuntz relations are satisfied.

Developing this example a bit further, we can see a relationship between Cuntz isometries and iterated function systems. Let C be the standard Cantor set in $[0,1]$, consisting of those real numbers whose ternary expansions are of the form $x = \sum_{k=1}^{\infty} \frac{x_k}{3^k}$ where $x_k \in \{0, 2\}$ for all k. Let

$$\varphi : C \to [0,1], \quad \varphi\left(\sum\nolimits_{k=1}^{\infty} \frac{x_k}{3^k}\right) = \sum\nolimits_{k=1}^{\infty} \frac{x_k}{2^{k+1}}.$$

Let m be Lebesgue measure on $[0,1]$, and define the Cantor measure μ on C by $\mu\left(\varphi^{-1}(B)\right) = m(B)$ if $B \subset [0,1]$ is Lebesgue measurable. This is well defined since φ is bijective except at countably many points.

Now define isometries R_0, R_1 on $\left(L^2(C), \mu\right)$ by defining their adjoints:

$$R_0^*(f) = S_0^*(f \circ \varphi) \quad \text{and} \quad R_1^*(f) = S_1^*(f \circ \varphi), \quad f \in \left(L^2(C), \mu\right).$$

Then

$$R_0^*(f)(x) = \frac{1}{\sqrt{2}} f\left(\frac{x}{3}\right) \quad \text{and} \quad R_1^*(f)(x) = \frac{1}{\sqrt{2}} f\left(\frac{x+2}{3}\right),$$

$f \in \left(L^2(C), \mu\right)$, $x \in C$. Thus we see the iterated function system for the Cantor set $\tau_0(x) = x/3$, $\tau_1(x) = (x+2)/3$ arising in the definition of Cuntz isometries on the Cantor set.

The Cuntz relations can be represented in many different ways. In their paper [DJ15a], Dutkay and Jorgensen look at finite Markov processes, and the infinite product of the state space is a compact set on which different measures can be defined, and these form the setting of representations of the Cuntz relations.

To construct a Fourier basis for a spectral measure arising from an iterated function system generated by contractions $\{\tau_0, \cdots, \tau_{N-1}\}$, Jorgensen (and others, [JP98, DPS14, DJ15a, PW17]) choose filters $m_0, \cdots, m_{N-1}$ and define Cuntz isometries $S_0, \cdots, S_{N-1}$ on $L^2(\mu)$ by

$$S_j f = \sqrt{N} m_j f \circ R,$$

where R is the common left inverse of the τ's. The filters, functions defined on the attractor set of the iterated function system, are typically chosen to be continuous, and are required to satisfy the relation $\sum_{j=0}^{N-1} |m_j|^2 = 1$. The Cuntz relations are satisfied by the S_j's provided the filters satisfy the orthogonality condition

$$\mathcal{M}^* \mathcal{M} = I, \quad (M)_{jk} = m_j(\tau_k(\cdot)). \tag{11}$$

To obtain Fourier bases, the filters m_j are chosen specifically to be exponential functions when possible. This is not possible in general, however, and is not possible in the case of the middle-third Cantor set and its corresponding measure μ_3.

The fact that some measures, such as μ_3, are not spectral leaves us with a conundrum: We still desire Fourier-type expansions of functions in $L^2(\mu)$, that is, a representation as a series of complex exponential functions, but we cannot get such an expansion from an orthogonal basis of exponentials in the case of a non-spectral measure. For this reason, we turn to another type of sequence called a frame, which has the same ability to produce series representations that an orthogonal basis does, but has redundancy that orthogonal bases lack and has no orthogonality requirement. Frames for Hilbert spaces were introduced by Dun and Schaeer [DS52] in their study of non-harmonic Fourier series. The idea then lay essentially dormant until Daubechies, Grossman, and Meyer reintroduced frames in [DGM86]. Frames are now pervasive in mathematics and engineering.

	Scaling factor	Number of affine maps τ_i	Ambient dimension	Hausdorff dimension
Middle-third C_3	3	2	1	$\log_3 2 = \frac{\ln 2}{\ln 3}$
The 4-ary C_4	4	2	1	$\frac{1}{2}$
Sierpinski triangle	2	3	2	$\log_2 3 = \frac{\ln 3}{\ln 2}$

Table 1: Some popular affine IFSs

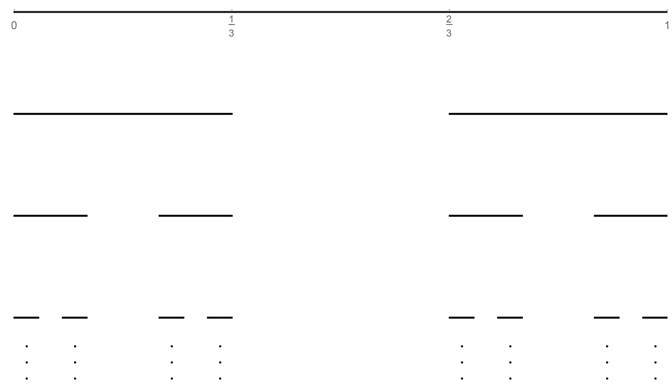

Figure 1: Middle-third Cantor C_3

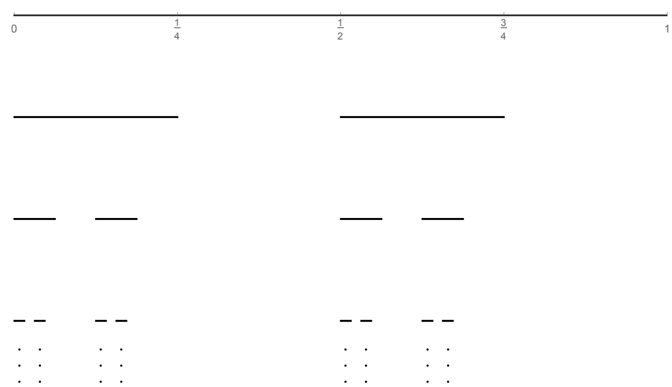

Figure 2: The 4-ary Cantor C_4

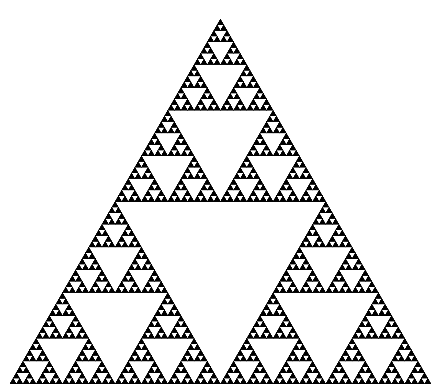

Figure 3: Sierpinski triangle

Frequency bands, filters, and representations of the Cuntz-algebras

Our analysis of the Cuntz relations here in the form $\{S_i\}_{i=0}^{N-1}$ turns out to be a modern version of the rule from signal-processing engineering (SPEE): When complex frequency response functions are introduced, the (SPEE) version of the Cuntz relations $S_i^* S_j = \delta_{ij} I_{\mathscr{H}}$, $\sum_{i=0}^{N-1} S_i S_i^* = I_{\mathscr{H}}$, where $\mathscr{H}$ is a Hilbert space of time/frequency signals, and where the N isometris S_i are expressed in the following form:

$$(S_i f)(z) = m_i(z) f\left(z^N\right), \ f \in \mathscr{H}, \ z \in \mathbb{C};$$

and where $\{m_i\}_{i=0}^{N-1}$ is a system of bandpass-filters, m_0 accounting for the low band, and the filters $m_i(z)$, $i > 0$, accounting for the remaining bands in the subdivision into a total of N bands. The diagram form (SPEE) is then as in Figure 4.

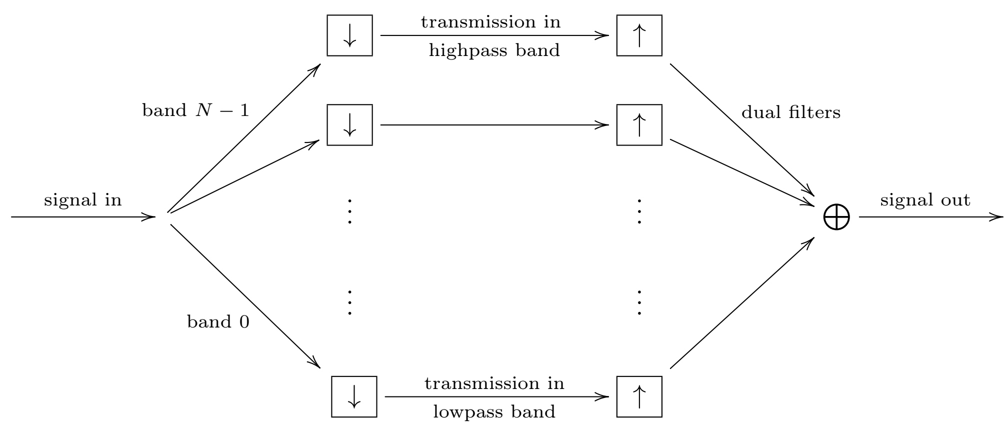

Figure 4: The picture is a modern math version of one I (PJ) remember from my early childhood: In our living room, my dad was putting together some of the early versions of low-pass/high-pass frequency band filters for transmitting speech signals over what was then long distance. One of the EE journals had a picture which is much like the one I reproduce here; after hazy memory. Strangely, the same multi-band constructions are still in use for modern wireless transmission, both speech and images. The down/up arrows in the figure stand for down-sampling, up-sampling, respectively. Both operations have easy expressions in the complex frequency domain. For example up-sampling becomes substitution of z^N where N is the fixed total number of bands.

Frames

Let $\mathscr{H}$ be a separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$, and let $\mathbb{J}$ be a countable index set. A frame for $\mathscr{H}$ is a sequence $\{x_j\}_{j \in \mathbb{J}} \subset \mathscr{H}$ such that there exist constants $0 < C_1 \leq C_2 < \infty$ such that for all $v \in \mathscr{H}$,

$$C_1 \|v\|^2 \leq \sum |\langle v, x_j \rangle|^2 \leq C_2 \|v\|^2 .$$

If C_1 and C_2 can be chosen so that $C_1 = C_2 = 1$, we say that $\{x_j\}$ is a *Parseval frame.*

If $\mathbb{X} \subset \mathscr{H}$ is a frame, then any other frame $\tilde{\mathbb{X}} := \{\tilde{x}_j\} \subset \mathscr{H}$ that satisfies

$$\sum \langle v, \tilde{x}_j \rangle x_j = v, \tag{12}$$

for all $v \in \mathscr{H}$ is called a *dual frame* for $\mathbb{X}$. Every frame possesses a dual frame, and in general, dual frames are not unique. A Parseval frame is self-dual, that is, $v = \sum \langle v, x_j \rangle x_j$.

Returning to our current interest, we say that a measure μ is frame-spectral if there exists a countable set $\Lambda \subset \mathbb{R}$ such that $\left\{e^{i2\pi\lambda x}\right\}_{\lambda\in\Lambda}$ is a frame in $L^2(\mu)$. In general, for a compact subset C of $\mathbb{R}^d$ with nonzero measure, Lebesgue measure restricted to that set is not spectral, but it will always be frame spectral. In general, a singular measure will not be frame spectral [DHSW11, DL14], but many singular measures are frame-spectral [EKW16, PW17]. It is currently unknown whether or not μ_3 is frame-spectral.

The redundancy of frames makes them more immune to error in transmission: Multiple frame elements will capture the same dimensions of information, and so if one series coefficient in the frame expansion of a function is transmitted incorrectly, the adverse effect on the reconstructed function will be minimal. However, expansions in terms of a given frame are in general not unique, and this can be a desirable or undesirable quality depending on the application. If we want the best of both worlds — a frame with redundancy but with a unique expansion for each function { then we must turn to the realm of Riesz bases.

A Riesz basis in a Hilbert space $\mathscr{H}$ is a sequence $\{x_j\}_{j=1}^{\infty}$ which has dense span in $\mathscr{H}$ and is such that there exist $0 < A \leq B$ such that for any nite sequence of scalars $c_1, c_2, \cdots, c_N$, we have

$$A \sum_{j=1}^{N} |c_j|^2 \leq \left\| \sum_{j=1}^{N} c_j x_j \right\|^2 \leq B \sum_{j=1}^{N} |c_j|^2 . \tag{13}$$

A Riesz basis is a frame that has only one dual frame. Equivalently, $\{x_j\}_{j=1}^{\infty}$ is a Riesz basis if an only if there is a topological isomorphism $T : \mathscr{H} \to \mathscr{H}$ such that $\{Tx_j\}_{j=1}^{\infty}$ is an orthonormal basis of $\mathscr{H}$.

The unit disk $\mathbb{D}$, for example, as a convex planar body has no orthogonal basis of complex exponential functions, but it does possess a frame of complex exponential functions. However, it is still an open problem whether it possesses a Riesz basis of complex exponential functions.

Description of Lectures

The lectures will cover the following overarching themes: the harmonic analysis of Cantor spaces (and measures) arising as fractals (including fractal dust) and iterated function systems (IFSs), as well as the methods used to study their harmonic analyses that span both the smooth and non-smooth domains. The topics that arise from these overarching themes include fractal Fourier analyses (fractals in the large), spectral theory of Ruelle operators; representation theory of Cuntz algebras; convergence of the cascade algorithm in wavelet theory; reproducing kernels and their boundary representations; Bernoulli convolutions and Markov processes. The remarkable aspect of these broad connections is that they often straddle both the smooth and non-smooth domains. This is particularly evident in Jorgensen's research on the cascade algorithm, as wavelets already possess a "dual" existence in the continuous and discrete worlds, and also his research on the boundary representations of reproducing kernels, as the non-smooth domains appear as boundaries of smooth domains. Another remarkable aspect is that these broad connections weave together to form a sharply focused theme.

The logical flow and interconnectedness of these topics are illustrated in Figure 5. The numbers in the figure correspond to the lecture numbers.

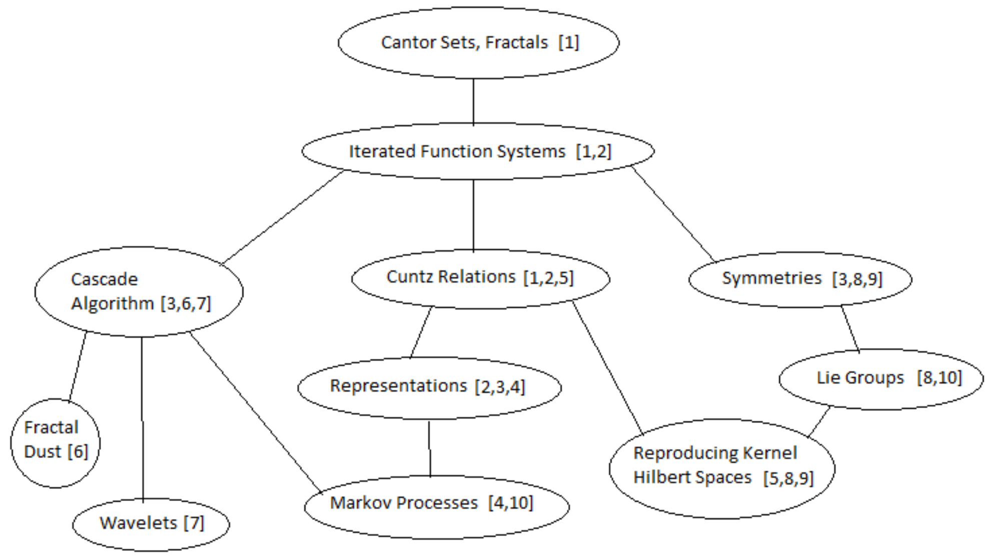

Figure 5: Flow and Connections of Topics.

Lecture 1. Harmonic Analysis of Measures: Analysis on Fractals

Beginning with the foundational results in "*Dense analytic subspaces in fractal L^2-spaces*" [JP98], the first talk will cover the construction of spectral measures, the constructions of various spectra, characterizations and invariance of spectra for spectral measures. The talk will include initial connections to representation theory of Cuntz algebras, spectra and tiling properties in $\mathbb{R}^d$, the Fuglede conjecture, and Reproducing Kernel Hilbert spaces.

The existence of orthogonal Fourier bases for classes of fractals came as somewhat of a surprise, referring to the 1998 Jorgensen-Pedersen paper. There are several reasons for why existence of orthogonal Fourier bases might have been unexpected: For one, existence of orthogonal Fourier bases, as in the classical case of Fourier, tends to imply a certain amount of "smoothness" which seems inconsistent with fractal geometries, and fractal dimension. Nonetheless, when feasible, such a orthogonal Fourier analysis holds out promise for applications to large chaotic systems, or to analysis of noisy signals; areas that had previously resisted analysis by Fourier tools.

When Fourier duality holds, it further yields a duality of scale, fractal scales in the small, and for the dual frequency domain, fractals in the large.

While the original framework for the Jorgensen-Pedersen fractals, and associated L^2-spaces, was a rather limited family, this original fractal framework for orthogonal Fourier bases has since been greatly expanded. While the original setting was restricted to that of affine selfsimilarity, determined by certain iterated affine function systems(IFSs) in one and higher dimension, this has now been broadened to the setting of say conformal selfsimilar IFS systems, and to associated maximal entropy measures. And even when the strict requirements entailed by orthogonal Fourier bases is suitably relaxed, there are computational Fourier expansions (Herr-Jorgensen-Weber) which lend themselves to analysis/synthesis for most singular measures.

Inherent in the study of fractal scales is the notion of multi-resolution analyses, in many ways parallel to the more familiar Daubechies wavelet multi-resolutions. Moreover, Strichartz proved that when an orthogonal Fourier expansions exist, they have localization properties which parallel the kind of localization which has made wavelet multi-resolutions so useful. The presence of multi-resolutions further implies powerful algorithms, and it makes connections to representation theory and to signal/image processing; subjects of the later lectures. Dutkay-Jorgensen proved that all affine IFS fractals have wavelet bases.

References and reading list. [JP98, JKS14a, JKS14c, Jor12, DJS12, DJ07]

Lecture 2. Spectra of measures, tilings, and wandering vectors

The second talk will build on the themes from the first talk, detailing the constructions of spectra arising from Cuntz algebras, characterizations of spectra using the spectral theory of Ruelle operators, connections between tilings, and wandering vectors for unitary groups and unitary systems.

There is an intimate relations between systems of tiling by translations on the one hand, and orthogonal Fourier bases on the other. Representation theory makes a link between the two, but the tile-spectral question is deep and difficult; so far only partially resolved. One tool of inquiry is that of "wandering vectors" or wandering subspaces. The term "wandering" has its origin in the study of systems of isometries in Hilbert space. It has come to refer to certain actions in a Hilbert space which carries representations: When the action generates orthogonal vectors, we refer to them as wandering vectors; similarly for closed subspaces. In the case of representations of groups, this has proved a useful way of generating orthogonal Fourier bases; — when they exist. In the case of representations of the Cuntz algebras, the "wandering" idea has become a tool for generating nested and orthogonal subspaces. The latter includes multiresolution subspaces for wavelet systems and for signal/image processing algorithms.

References and reading list. [Kad16, IK13, DJ15b, DJ15c, DJ11, She15]

Lecture 3. The universal tiling conjecture in dimension one and operator fractals

The third talk will focus on the tiling properties arising from the study of spectral measures, specifically in dimension one; advances in the Fuglede conjecture in dimension one, non-commutative fractal analogues in infinite dimensions.

Fuglede (1974) conjectured that a domain Ω admits an operator spectrum (has an orthogonal Fourier basis) if and only if it is possible to tile $\mathbb{R}^d$ by a set of translates of Ω [Fug74]. Fuglede proved the conjecture in the special case that the tiling set or the spectrum are lattice subsets of $\mathbb{R}^d$ and Iosevich et al. [IKT01] proved that no smooth symmetric convex body Ω with at least one point of nonvanishing Gaussian curvature can admit an orthogonal basis of exponentials.

Using complex Hadamard matrices of orders 6 and 12, Tao [Tao04] constructed counterexamples to the conjecture in some small Abelian groups, and lifted these to counterexamples in $\mathbb{R}^5$ or $\mathbb{R}^{11}$. Tao's results were extended to lower dimensions, down to $d = 3$, but the problem is still open for $d = 1$ and $d = 2$.

Summary of some affirmative recent results: The conjecture has been proved in a great number of special cases (e.g., all convex planar bodies) and remains an open problem in small dimensions. For example, it has been shown in dimension 1 that a nice algebraic characterization of finite sets tiling $\mathbb{Z}$ indeed implies one side of Fuglede's conjecture [CM99]. Furthermore, it is sufficient to prove these conditions when the tiling gives a factorization of a non-Hajós cyclic group [Ami05].

Ironically, despite a large number of great advances in the area, Fuglede's original question is still unsolved in the planar case. In the planar case, the Question is: Let Ω be a bounded open and connected subset of $\mathbb{R}^2$, does it follow that $L^2(\Omega)$ with respect to planar Lebesgue measure has an orthogonal Fourier basis if and only if Ω tiles $\mathbb{R}^2$ with translations by some set of vectors from $\mathbb{R}^2$. Of course, if Ω is a fundamental domain for some rank-2 lattice, the answer is affirmative on account of early work.

Another direction is to restrict the class of sets Ω in $\mathbb{R}^3$ to be studied. One such recent direction is the following affirmative theorem for the case when Ω is assumed to be a convex polytope: Nir Lev et al [GL17] proved that a spectral convex polytope (i.e., having a Fourier basis) must tile by translations. This implies in particular that Fuglede's conjecture holds true for convex polytopes in $\mathbb{R}^3$.

References and reading list. [JKS14b, DJ13a, DJ13b, DHJP13]

Lecture 4. Representations of Cuntz algebras associated to quasi-stationary Markov measures

The fourth talk concerns representations of Cuntz algebras that arise from the action of stochastic matrices on sequences from $\mathbb{Z}_n$. This action gives rise to an invariant measure, which depending on the choice of stochastic matrices, may satisfy a finite tracial condition. If so, the measure is ergodic under the action of the shift on the sequence space, and thus yields a representation of a Cuntz algebra. The measure provides spectral information about the representation in that equivalent representations of the Cuntz algebras for different choices of stochastic matrices occur precisely when the measures satisfy a certain equivalence condition.

Recursive multiresolutions and basis constructions in Hilbert spaces are key tools in analysis of fractals and of iterated function systems in dynamics: Use of multiresolutions, selfsimilarity, and locality, yield much better pointwise approximations than is possible with traditional Fourier bases. The approach here will be via representations of the Cuntz algebras. It is motivated by applications to an analysis of frequency sub-bands in signal or image-processing, and associated multi-band filters: With the representations, one builds recursive subdivisions of signals into frequency bands.

Concrete realizations are presented of a class of explicit representations. Starting with Hilbert spaces $\mathscr{H}$, the representations produce recursive families of closed subspaces (projections) in $\mathscr{H}$, in such a way that "non-overlapping, or uncorrelated, frequency bands" correspond to orthogonal subspaces in $\mathscr{H}$. Since different frequency bands must exhaust the range for signals in the entire system, one looks for orthogonal projections which add to the identity operator in $\mathscr{H}$. Representations of Cuntz algebras achieve precisely this: From representations we obtain classification of families of multi-band filters; and representations allow us to deal with non-commutativity as it appears in both time/frequency analysis, and in scale-similarity. The representations further offer canonical selections of special families of commuting orthogonal projections.

References and reading list. [DJ15a, DHJ15]

Lecture 5. The Cuntz relations and kernel decompositions

The fifth talk concerns representations of Cuntz algebras and their relationship to harmonic analysis of measures, particularly singular measures. As will be demonstrated in earlier talks, the constructions of spectral measures often utilize "Cuntz isometries", namely isometries that satisfy the Cuntz relations. This talk will discuss how understanding specific representations of the Cuntz algebras yields information concerning other spectra for a spectral measure. Conversely, beginning with a representation of a Cuntz algebra, a Markov measure can be associated to the representation which gives spectral information about the representation.

References and reading list. [DJ14, DJ12b, DJ12a]

Lecture 6. Harmonic analysis of wavelet filters: input-output and state-space models

The sixth talk will focus on the connections between harmonic analysis on fractals and the cascade algorithm from wavelet theory. Wavelets have a dual existence between the discrete and continuous realms manifested in the discrete and continuous wavelet transforms. Wavelet filters give another bridge between the smooth and non-smooth domains in that the convergence of the cascade algorithm yields wavelets and wavelet transforms in a smooth setting, i.e. $\mathbb{R}^d$, and also the non-smooth setting such as the Cantor dust, depending on the parameters embedded in the choice of wavelet filters.

References and reading list. [AJL16, AJL15, AJLM13, BJMP05]

Lecture 7. Spectral theory for Gaussian processes: reproducing kernels, boundaries, and L^2-wavelet generators with fractional scales

The seventh talk concerns Gaussian processes for whose spectral (meaning generating) measure is spectral (meaning possesses orthogonal Fourier bases). These Gaussian processes admit an Itô-like stochastic integration as well as harmonic and wavelet analyses of related Reproducing Kernel Hilbert Spaces.

References and reading list. [AJK15, AJ12, AJL11, JS09, DJ06a]

Lecture 8. Reproducing Kernel Hilbert Spaces arising from groups

The eighth talk concerns Reproducing Kernel Hilbert Spaces that appear in the study of spectral measures. Spectral measures give rise to positive definite functions via the Fourier transform. Reversing this process will be the focus of the ninth talk. This talk will set the stage by discussing Reproducing Kernel Hilbert Spaces that appear in the context of positive definite functions, and the harmonic analysis of those Reproducing Kernel Hilbert Spaces.

References and reading list. [JPT16, DJ09, DJ08, DJ06b]

Lecture 9. Extensions of positive definite functions

The ninth talk will consider the question of spectral measures from the perspective of positive definite functions. Since the measures are spectral, the corresponding positive definite functions have special properties in terms of their zero sets. This correspondence leads to the natural question of whether this process can be reversed. Bochner's theorem implies that positive definite functions are the Fourier transform of measures, but whether those measures are spectral becomes a subtle problem. Thus, by considering certain functions on appropriate subsets, the question of spectrality can be formulated as whether the function can be extended to a positive definite function. The answer is sometimes yes, using the harmonic analysis of Reproducing Kernel Hilbert Spaces.

References and reading list. [JT15, JP10, DJ10, CBG16]

Lecture 10. Reflection positive stochastic processes indexed by Lie groups

The tenth talk will focus on stochastic processes that appear in the representation theory of Lie groups. Motivated by reflection symmetries in Lie groups, this talk will consider representation theoretic aspects of reflection positivity by discussing reflection positive Markov processes indexed by Lie groups, measures on path spaces, and invariant Gaussian measures in spaces of distribution vectors. This provides new constructions of reflection positive unitary representations.

Since early work in mathematical physics, starting in the 1970ties, and initiated by A. Jaffe, and by K. Osterwalder and R. Schrader, the subject of reflection positivity has had an increasing influence on both non-commutative harmonic analysis, and on duality theories for spectrum and geometry. In its original form, the Osterwalder-Schrader idea served to link Euclidean field theory to relativistic quantum field theory. It has been remarkably successful; especially in view of the abelian property of the Euclidean setting, contrasted with the non-commutativity of quantum fields. Osterwalder-Schrader and reflection positivity have also become a powerful tool in the theory of unitary representations of Lie groups. Co-authors in this subject include G. Olafsson, and K.-H. Neeb.

The topics will be outlined in lecture 10, and there has been recent Oberwolfach conferences in the field, where Jorgensen has played a leading role.

References and reading List. [JNO16, She15]

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