Ph.D. Qualifying examination in topology

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Instructions. Do eight problems, four from each part. Some problems may require ideas from both semesters, and some problems may go beyond what was covered in the course. This is a closed book examination. You should have no books or papers of your own. Please do your work on the paper provided. Clearly number your pages to correspond with the problem you are working. You may use "big theorems" (i.e. Urysohn's Lemma, Sard) provided that the point of the problem is not the proof of the theorem. Always justify your answers.

Please indicate here which eight problems you want to have graded:

A1 A2 A3 A4 A5 A6 B1 B2 B3 B4 B5 B6

Notation: R^n is Euclidean n-space, with the usual topology and differentiable structure.

S^n is the n-sphere, the set of points distance one from the origin in R^{n+1}, with the subspace topology, and with the usual differentiable structure.

"Manifold" means compact differentiable manifold without boundary, unless otherwise noted.

Where appropriate, always assume usual product, quotient or subspace topologies. Regular and normal spaces are required to be T_1.

1. **Part A**

A1) Prove or give a counterexample: The product of two regular spaces is regular.

A2) Define the uniform and box topologies on a product of topological spaces. Let $X = R^J$ be the product of a countable number of copies of the real numbers. Prove that the product, uniform and box topologies yield three distinct, non-homeomorphic topologies on X.

A3) Let X be $S^2 - \{(0, 0, \pm 1)\}$, that is X is the result of removing the north and south poles from the unit sphere. Define two points of X to be equivalent if and only if they lie on the same great circle through the North and South poles. Identify the quotient space of these equivalence classes, giving an explicit homeomorphism.

A4) Prove that a subspace of a second countable space is second countable. Give an example of a metric space which is not second countable.

A5) Suppose $A = \cup A_\alpha$, where each A_α is connected, and so that there is a point x common to all A_α. Prove that A is connected.

A6) Let Y be a compact Hausdorff space, and let $f : X \to Y$ be a function. Recall that the graph G_f of f is

$$G_f = \{(x, f(x)) \in X \times Y\}$$

Prove that f is continuous if and only if the graph G_f of f is closed in $X \times Y$.

2. **Part B**

B1) Let M be a connected smooth manifold. Prove that the DeRham cohomology group $H^0(M) = R$.

B2) Suppose M is a Lie group. Sketch the proof that M is parallelizable.

B3) Let $W_c = \{(x, y, z, w) \in R^4 : xyz = c\}$ and $Y_c = \{(x, y, z, w) \in R^4 : xzw = c\}$. For what real numbers c is Y_c a three-manifold? For what pairs (c_1, c_2) is $W_{c_1} \cap Y_{c_2}$ a two-manifold.

B4) Consider the one form

$$\omega = \frac{-ydx + xdy}{x^2 + y^2}$$

on $R^2 \setminus (0, 0)$.

a) Prove that $d\omega = 0$.

b) Let σ be the pullback of ω to the unit circle $S^1 \subset R^2$. Compute

$$\int \sigma$$

over S^1, where S^1 is oriented counterclockwise. Tell why these facts imply that the DeRham cohomology $H^1(S^1) \neq 0$.

B5) Give the definition of the tangent space T_pM to a manifold M at a point p, in terms of derivations acting on germs. Discuss why one may think of a tangent vector as a velocity vector of a curve on a manifold, and how this relates to the geometric tangent plane of a submanifold of R^3. Explain the differential map df in these three settings. Be precise.

B6) Define the notion of a smooth action of a Lie group G on a smooth manifold M. Give an example of S^1 acting smoothly on S^2. Prove that in general an action of S^1 yields a flow on M and therefore a vector field on M. Must this vector field be never zero?