Qualifying Exam in Topology

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This is a three hour exam. You are not allowed to use any written aids like notes during this exam. Work four questions from part 1 and four questions from part 2. Indicate on the front page of your exam which problems you want graded. You may use results about functions in euclidean space (chain rule, implicit or inverse function theorems for example) or "big theorems" (such as Sard's theorem, the regular value theorem) provided you give the precise statement.

1 General Topology

1. Suppose that X and Y are connected, nonempty topological spaces. Prove that $X \times Y$ is connected.

2. Suppose that X is a compact Hausdorff space and A and B are closed disjoint subsets of X. Prove that there exists U and V open in X with $U \cap V = \emptyset$ and $A \subset U$, $B \subset V$. That is, prove that X is normal.

3. Suppose that X is Hausdorff and $f, g : Y \to X$ are continuous. Prove that the set $\{y \in Y | f(y) = g(y)\}$ is closed.

4. A space is *separable* if it has a countable dense subset. Prove that a topological space is separable if it is second countable, and that a metric space is second countable if it is separable. *There are two things to prove here.*

5. Prove that if X is compact, Y is Hausdorff and $f : X \to Y$ is one to one, onto and continuous, then f is a homeomorphism.

6. Suppose that $p : X \to Z$ is a quotient map and $f : X \to Y$ is continuous. Prove that there exists a continuous map $\overline{f} : Z \to Y$ with $\overline{f} \circ p = f$ if and only if for every x_1, x_2 with $p(x_1) = p(x_2)$ it is the case that $f(x_1) = f(x_2)$.

2 Differential Topology

1. Prove the local immersion theorem. If $F : M \to N$ is a smooth map of smooth manifolds and dF (the differential of F) is injective at P then there are local coordinates (U, ϕ) at P and (V, ψ) at $F(P)$ so that $F(U) \subset V$ and,

$$\psi \circ F \circ \phi^{-1}(x^1, \ldots, x^m) = (x^1, \ldots, x^m, 0, \ldots, 0).$$

2. Let $O(n)$ denote the set of $n \times n$ matrices with real entries, A so that $AA^t = Id$. Prove that $O(n)$ is a Lie group, and identify the tangent space at the identity.

3. Let T^2 be the subset $\mathbb{R}^3$ that is the result of rotating the circle in the yz plane of radius 1 centered at $(0, 2, 0)$ about the z-axis. It may be useful to note that T^2 is the set of points in 3-space satisfying the equation $((x^2 + y^2)^{1/2} - 2)^2 + z^2 = 1$.

   - Prove that T^2 is a smooth 2-manifold.
   - Let $p : T^2 \to \mathbb{R}^2$ be the restriction of $p : \mathbb{R}^3 \to \mathbb{R}^2$ given by $p(x, y, z) = (x, y)$. Identify the regular values of $p : T^2 \to \mathbb{R}^2$.

4. Let S^2 be the unit sphere centered at the origin in $\mathbb{R}^3$ given the standard orientation as the boundary of the ball. Let ω be the two form on S^2 that is the restriction of $xdy \wedge dz - ydx \wedge dz + zdx \wedge dy$ to the sphere. Compute

$$\int_{S^2} \omega.$$

5. Let M be a smooth manifold.

   - Given $P \in M$ define $T_P M$.
   - Define smooth vector field on M.

- If X and Y are smooth vector fields on M define $[X, Y]$.
- Compute $[X, Y]$ where $X = x\frac{\partial}{\partial y} + \frac{\partial}{\partial x}$ and $Y = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}$ are vector fields on the plane.

6. Suppose that M and N are smooth manifolds. Give $M \times N$ the structure of a smooth manifold by producing a compatible atlas. Prove that your atlas is compatible.