Ph.D. Qualifying Exam in Topology

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January 20, 2008

Instructions. Do eight problems, four from each part. Some problems may require ideas from both semesters 22M:132-22M:133, and some problems may go beyond what was covered in the course. This is a closed book examination. You should have no books or papers of your own. Please do your work on the paper provided. Clearly number your pages to correspond with the problem you are working. When you start a new problem, start a new page; and please write only on one side of the paper.
You may use "big theorems" provided that the point of the problem is not the proof of the theorem.

Always justify your answers unless explicitly instructed otherwise.

Please indicate here which eight problems you want to have graded:

A1 A2 A3 A4 A5 A6

B1 B2 B3 B4 B5 B6

Notation:
$\mathbb{R}^n$ is Euclidean n-space, with the usual topology and differentiable structure.

S^n is the n-sphere, the set of points distance one from the origin in $\mathbb{R}^{n+1}$, with the subspace topology, and with the usual differentiable structure.

Part A

A1) Suppose that X and Y are connected, prove that $X \times Y$ is connected.

A2) **Tube Lemma** Suppose that X is compact and $X \times \{y\} \subset U$ where $U \subset X \times Y$ is open. Prove that there exists $W \subset Y$ open so that $X \times \{y\} \subset X \times W \subset U$.

A3) Suppose that X is compact and Y is Hausdorff. Prove that if $f : X \to Y$ is continuous then $f(X)$ is closed. This proof should be from definitions.

A4) Let $\mathbb{R}P(n)$ be the quotient space from $\mathbb{R}^{n+1} - \{\vec{0}\}$ by the equivalence relation $(x_0, \dots, x_n) \sim (x'_0, \dots, x'_n)$ if there exists $\lambda \in \mathbb{R}$ so that

$$\lambda(x_0, \dots, x_n) = (x'_0, \dots, x'_n).$$

a. Prove that the quotient map $q : \mathbb{R}^{n+1} - \{\vec{0}\} \to \mathbb{R}P(n)$ is open.
b. Prove that $\mathbb{R}P(n)$ is second countable.

A5) Let X be a topological space and let

$$\Delta = \{(x,x) \in X \times X\}.$$

Prove that X is Hausdorff if and only if $\Delta \subset X \times X$ is closed.

A6) Prove that if X is compact, Y is Hausdorff, and $f : X \to Y$ is continuous, one-to-one and onto then f is a homeomorphism.

Part B

B1) Prove that the sphere $S^2 = \{(x, y, z) \in \mathbb{R}^3 \ : \ x^2 + y^2 + z^2 = 1\}$ is a smooth manifold by exhibiting charts. Carefully calculate <u>one</u> transition map and its derivative to explain why the transition map is a diffeomorphism.

B2) Prove that the surface $H = \{(x, y, z) \in \mathbb{R}^3 \ : \ x^2 - y^2 + z^2 = 3\}$ is a smooth manifold by showing H is the pre-image of a regular value of a smooth map. Justify your various claims.

B3) Let $A = \{(x, y, z) \in \mathbb{R}^3 \ : \ z = 2x + 3y\}$ and let B =the $z-$ axis.
Prove A and B meet transversally.

B4) Suppose X, Y, Z are smooth manifolds, $f : X \to Y$ is a diffeomorphism, and $g : Y \to Z$ is a smooth map such that for some $y \in Y$, $dg_y : T_yY \to T_{g(y)}Z$ is injective. Prove that for each $x \in f^{-1}(y)$, $d_x(g \circ f)$ is injective.

B5) Let $\mathcal{M}$ be the set of all 2×2 matrices (with real-number entries).

a. Let $\mathcal{M}_0 = \{M \in \mathcal{M} \ : \ det(M) \neq 0\}$. Explain how we can view $\mathcal{M}_0$ as a smooth manifold. Make sure to explain the dimension of $\mathcal{M}_0$.
b. Let $\mathcal{M}_1 = \{M \in \mathcal{M} \ : \ det(M) = 1\}$. Explain how we can view $\mathcal{M}_1$ as a smooth manifold. Make sure to explain the dimension of $\mathcal{M}_1$.

B6) Consider the following 1-form on $\mathbb{R}^3$.

$$\omega = xy\, dx + x\, dy + x\, dz$$

(i) Calculate the 2-form $d\omega$.
(ii) Show by explicit calculation that $d(d\omega) = 0$.
(iii) Does there exist a 1-form α on $\mathbb{R}^3$ such that $dd\alpha = (x^5y^3z^9)dx \wedge dy \wedge dz$? (Do not try to find such α; just state in one or two sentences why such a 1-form does or does not exist.)